Introduction to Feynman Diagrams

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Abstract

In this paper, Feynman diagrams are presented as depictions of particle paths through spacetime. This is done in the context of the fourth-order anharmonic modification of the free field theory. After presenting the rules that relate a Feynman diagram to its corresponding mathematical term, we provide a glimpse of the importance of Green's functions in this context. To conclude the paper, we prove the logarithm property of the generating functional, which shows a deep relation between connected and disconnected diagrams.

Introduction

In Quantum Field Theory, Feynman diagrams provide a visual representation of terms in the series expansion of probability amplitude quantities. Equivalently, they illustrate how particles appear and, after propagating for some distance and possibly interacting with other particles, disappear. We will introduce Feynman diagrams from the formalism of path integrals. Of course, the specific relation between the physical quantities appearing in the path integrals and the features characterizing the corresponding Feynman diagrams will be explained and explored. The work presented is based on chapter I.7 from Zee's *Quantum Theory in a Nutshell* (2010, pp. 43-55).

From separated to intertwined paths

Feynman diagrams are most easily understood through a particular example. It is especially convenient to consider an anharmonic modification of the free field or Gaussian theory. In the pure free field model, the equation of motion is linear. This impllies that two independent fields coinciding in space, φ_1 and φ_2, will propagate without affecting each other. This is, each mode of vibration behaves as if the other were not present at all. Now, with the purpose of studying interaction between different solutions of our theory –which is the mathematical requirement for our theory to include collisions between particles-, we add the anharmonic potential term $-\frac{\lambda}{4!}\varphi^4$ to the free field Lagrangian.

As usual, let J(x) represent the source function, which indicates the locations in spacetime of sources and sinks of particles, φ(x) be the field, and m be the characteristic mass of the particles studied –or, alternatively, the mass appearing in the free field Lagrangian. Then, path integral formulation of Quantum Field Theory dictates that, to calculate the probability amplitude $Z(J)$ for a given displacement, it is enough to evaluate the corresponding integral:

$$Z(J) = \int D\varphi \, e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4 + J\varphi\}} \quad [1]$$

Where the λ dependence of Z is suppressed; this is, the scattering amplitude λ is fixed, so that Z is regarded a function only of J.

Integrating in series

This section is devoted to the step-by-step computation of Eq. [1]. After all, the charm of the path integral formulation is that it reduces the prediction of physical outcomes to the evaluation of integrals, like the one at hand. Of course, this is easier said than done.

In this case, the main trick is to express the exponential in Eq. [1] as the product of simpler exponential terms, which are then expanded by means of an infinite Taylor series. More concretely, we begin with the following manipulations:

$$e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4+J\varphi\}} = e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\}}e^{i\int d^4x\{-\frac{\lambda}{4!}\varphi^4\}}$$

$$= e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\}}\cdot[1-\frac{i\lambda}{4!}\int d^4x\,\varphi^4-\frac{1}{2}\left(\frac{i\lambda}{4!}\right)^2(\int d^4x\,\varphi^4)^2+\cdots]$$

After applying this expansion to the integrand of Eq. [1], we are left with the sum of infinitely many integrals that can be evaluated one by one:

$$Z(J) = \int D\varphi\, e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\}}[1-\frac{i\lambda}{4!}\int d^4x\,\varphi^4+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2(\int d^4x\,\varphi^4)^2-\cdots] \quad [2]$$

Look closely to the particular term $\int D\varphi\, e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\}}\cdot(\int d^4x\,\varphi^4)$. Note it which can be written as $\left(\int d^4\,w\left[\frac{\delta}{i\delta J(w)}\right]^4\right)\int D\varphi\, e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\right\}}$, since each variational derivative with respect to $J(w)$ brings down one φ from the exponent. Then, Eq. [2] becomes:

$$Z(J) = \left[1-\frac{i\lambda}{4!}\left(\int d^4\,w\left[\frac{\delta}{i\delta J(w)}\right]^4\right)^2+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\int d^4\,w\left[\frac{\delta}{i\delta J(w)}\right]^4\right)^3-\cdots\right]$$

$$\int D\varphi\, e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\right\}} =$$

$$e^{\frac{i\lambda}{4!}\int d^4w\left[\frac{\delta}{i\delta J(w)}\right]^4}\int D\varphi\, e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]+J\varphi\right\}} \quad [3]$$

Undoing in the last step the series expansion to recover the exponential form of the λ term. Just like in a magic trick, the λ dependence of Z was decomposed and then rebuilt outside the integral. At this point, the remaining integral in Eq. [3] looks conveniently familiar. It can be considered a generalized case of a Gaussian integral, of the kind introduced step by step (from the scalar to the mattress to the continuous model) in Zee's chapter I.3 (2010, pp.17-24). In fact, equation number 18 in that chapter gives its explicit value, which we now substitute in Eq. [3]:

$$Z(J) = Z(0,0)e^{\frac{i\lambda}{4!}\int d^4w\left[\frac{\delta}{i\delta J(w)}\right]^4} e^{-\left(\frac{i}{2}\right)\int\int d^4x d^4y\, J(x)D(x-y)J(y)} \quad [4]$$

Where the overall factor $Z(0,0) \equiv Z(J=0, \lambda=0)$ is not important for our current purposes and $D(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-m^2+i\varepsilon}$ is the propagator function. In a general d-dimensional spacetime, the factor $\frac{d^dk}{(2\pi)^d}$ would replace $\frac{d^4k}{(2\pi)^4}$. For the derivation of this result, check Zee's chapter I.3 (2010, pp.17-24). Finding Eq. [4] was our only objective until now. However, the obtained result requires itself a suitable interpretation in physical terms, which will be the topic of the remaining of this paper.

Series terms as Feynman Diagrams

We have just experienced how useful can be the correspondence between exponentials and their equivalent Taylor series. In particular, from Eq. [4], $Z(J)$ can be expressed as a double series expansion in λ and the double integral whose integrand goes as J^2; for convenience I will use the terminology "the term goes as J^k" to mean, specifically, that said integral in that term contains the product of *k* different J factors: $J(x_1) \cdot \cdots \cdot J(x_k)$. As can be checked directly, the net effect of the operator $\left[\frac{\delta}{i\delta J(w)}\right]^k$ when it acts on a J^l integral is to reduce the number of J factors to $l-k$ (if $l \geq k$) or bringing the integral to zero (if $l < k$). Just like in an ordinary k'th derivative with respect to x applied to a term proportional to x^l. Therefore, the double expansion of $Z(J)$ mentioned earlier contains a term going as $\lambda^n J^{2m-4n}$, for all $n,\ m = 0, 1, 2$ ... such that $2m \leq 4n$. To put it differently, we have products of a λ exponential and a J exponential; the exponents of λ can be any integers greater or equal to 0, while those of J are restricted to even integers also starting at 0. All combinations consistent with these restrictions are present in the series; take $\lambda^1 J^2$ or $\lambda^5 J^{20}$ as unpretentious examples.

Without further introduction, we are now ready to understand Feynman diagrams. They are, in short, pictures or schemes that conveniently represent terms in the double expansion of $Z(J)$. However, it is important to keep in mind that, because of the direct connection between the quantities appearing in Eq. [4] and physical quantities, Feynman diagrams express in turn concrete processes, like the collision between two or more given particles. What's more, the diagrams can be used to calculate the probability amplitude of such processes, as we will see with more detail in the next section. For now, we simply present the basic rules to be followed when associating diagrams with terms in the expansion of Z.

For a term of the form $-\lambda^n J^{2m-4n}$, the corresponding Feynman diagram: (1) is made of lines and vertices at which four lines meet; (2) has n vertices; (3) has m lines; and (4) presents $2m - 4n$ external ends. (Zee. 2010, p.45).

Remember: the introduction to the path integral formulation showed that $J(x)$ is related to sources and sinks of the field φ, which are simply particles that interact with φ. This insight allows us to interpret rule number (4) in physical terms: the external ends or lines stand for incoming or outcoming particles in the physical process studied. An important remark, note that here the scattering φ + φ → φ + φ is counted as a process involving four particles; in general, the total number of particles is the sum of the incoming and the outcoming ones. Furthering the correspondence, lines reproduce of the trajectories of particles through spacetime, and vertices (which correspond to the power of λ) are the expression of the collisions where more than one particle coincide. In particular, note how the introduction of the anharmonic factor λ led to the presence of particle scattering. In the pure free field theory, there are no λ and all the terms in $Z(J)$ correspond to diagrams in which particles can propagate independently, but not affect each other or, in other words, diagrams constituted of straight, unconnected lines. It is clear then why the introduction of λ was needed to make things more interesting, and why it is usually referred to as "the coupling constant".

Finally, we understand how these funny diagrams full of wiggles and crossings represent interactions between real (and/or not so real) particles in the universe. The next section will elaborate on this relation.

Green’s functions and collisions of particles

For many purposes, it is convenient to express $Z(J)$ as a power series in J. This is done directly by expanding Eq. [4] in the double series form, applying the variational derivatives to the J-depending terms, and rearranging the resulting terms in groups by common power of J. In this way, Eq. [4] becomes:

$$Z(J) = \mathrm{Z}(0,0)\sum_{s=0}^{\infty}\frac{i^s}{s!}\int dx_1\cdot\cdots\cdot\; dx_s\, J(x_1)\cdot\cdots$$

$$\cdot\, J(x_s)\int D\varphi\, e^{i\int d^4x\left\{\frac{1}{2}[(\partial\varphi)^2-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4\right\}}\varphi(x_1)\cdots\varphi(x_s)$$

$$\equiv\; \mathrm{Z}(0,0)\sum_{s=0}^{\infty}\frac{i^s}{s!}\int dx_1\cdot\cdots\cdot\; dx_s J(x_1)\cdot\cdots\cdot\; J(x_s)\, G^{(s)}(x_1,\cdots,x_s)\quad [5]$$

Note that the above can be interpreted as a rearrangement of the Feynman diagrams in order of increasing number of external ends, because this number is proportional to the power of J. The function $G^{(s)}$ just defined is called the s-point Green’s function. It follows from the rules in the last section that $G^{(s)}$ is closely related to diagrams including a total of s external ends. Consider again the connection between $J(x)$ and physical particles, and note how the integral of the s’th term in the series above contains s different $J(x)$ terms and hence depends on s particles. This implies in turns that the probability amplitude of an interaction (or, in concrete terms, a collision) between s given particles is directly proportional to $G^{(s)}$, and in fact completely determined by it (up to the normalization constant). So the conclusion is that Green’s functions represent the propagation of possibly interacting particles from some initial locations to some other points in spacetime. For instance, $G(x_1, x_2)$ describes the propagation of a single particle from x_1 to x_2, while $G(x_1, x_2, x_3, x_4)$ represents the scattering of two particles that start in x_1, x_2 and end up in x_3, x_4. Therefore, by translational invariance, $G^{(s)}(x_1,\cdots,x_s)$ only depends on the differences between the arguments, $x_i - x_j$ (i, j in [1, s], $i \neq j$), and not directly on the individual x_i, because the probability of these kind of processes must obviously be a function of the distances between the points rather than the points themselves.

To sum up, Feynman diagrams are a convenient representation of terms in the expansion of $\mathrm{Z(J)}$ and, at the same time, depictions of the trajectories and interactions undergone by particles that travel through spacetime. In addition, they allow us to calculate the probability amplitude related to each physical process in the most straightforward way known. Several examples, such as the collision between two mesons, are worked out in great detail in chapter I.7 of Zee's textbook (2010, pp. 51-57).

Disconnected graphs from connected components

In this last section, we are interested in characterizing the connectedness of Feynman diagrams. A graph is said to be connected if any two vertices in the graph are linked by a sequence of one or more lines, which is called a path. If this is not true, the graph is disconnected, and clearly formed by more than one disconnected pieces. In order to determine which diagrams are connected, we rewrite Eq. [4] in yet another form:

$$\mathrm{Z}(\mathrm{J}, \lambda) = \mathrm{Z}(\mathrm{J} = 0, \lambda) e^{W(J,\lambda)} = \mathrm{Z}(\mathrm{J} = 0, \lambda) \sum_{N=0}^{\infty} \frac{(W(J, \lambda))^N}{N!} \quad [6]$$

This is simply a way of distributing the terms in two disjoint groups, determined by the presence or absence of any power of J, that are multiplied together. Note that Eq. [6] constitutes itself the definition of $W(J,\lambda)$. In this sense, the terms encompassed by $\mathrm{Z}(\mathrm{J}=0,\lambda)$ are related, by definition, to diagrams with no J dependence and hence, by rule (4), no external legs. These diagrams are constituted by one or several connected loops that close on themselves. In contrast, all diagrams that are part of $W(J,\lambda)$ must all include a number of external sources.

Now, we prove that Z is a set composed by connected and disconnected diagrams, while $W(J,\lambda)$ contains only connected diagrams. This derivation is based on Srednicki (2007, pp. 74-76). To begin with, we know that diagrams represent term*s* in the series. So each different addend of the infinite sum of [6] corresponds to an independent diagram. That is why, in essence, Z is a set of many diagrams, rather than a single, complex diagram built from (infinitely) many pieces. $W(J,\lambda)$ is formed by a sum of several terms too, so the same applies to it. Alternatively, multiplying two terms together renders another term whose powers of J and λ are the sum of the powers of the original factors. Therefore, the multiplication operation corresponds to merging two diagrams into a new one in any way such that the total number of lines, vertices, and of external lines is conserved. So we see that the $(W(J,\lambda))^N$ term in the Taylor series of Eq. [6] represents all possible diagrams constructed from combinations of N isolated diagrams of W, repetition allowed. Now, by definition, $\mathrm{Z}(\mathrm{J})$ is proportional (up to a normalization constant) to the sum (or the set) of all possible diagrams. Express this by:

$$\mathrm{Z}(\mathrm{J}) \propto \sum_{n_i} D_{n_i} \; [7]$$

We already mentioned that any diagram can be decomposed into its connected factors. This is:

$$D_{n_i} = \frac{1}{S_D}\prod_i (C_i)^{n_i} \ [8]$$

Where S_D is a symmetry normalization factor which accounts for the number of different combinations of times C_is that can be combined to obtain D_{n_i}. For n_i identical C_i factors, there exist $n_i!$ identical rearrangements. So overall, counting over all i, we find:

$$S_D = \prod_i n_i! \ [9]$$

And the last three equations together render:

$$\mathrm{Z(J)} \propto \sum_{n_i} D_{n_i} \propto \sum_{n_i}\prod_i \frac{1}{n_i!}(C_i)^{n_i} \propto \prod_i \sum_{n_i}\frac{1}{n_i!}(C_i)^{n_i} = \prod_i e^{C_i} = e^{\sum_i C_i} \ [10]$$

So Z is proportional to the exponential of all connected diagrams. But, by Eq. [6], Z is also proportional to the exponential of $W(J,\lambda)$. It makes sense to impose the normalization constraint $\mathrm{Z}(\mathrm{J}=0,\lambda)$ = 1 –which amounts, in physical terms, to disregard the diagrams with no sources, called vacuum diagrams. This sets the proportionality constant in Eq. [7], and hence in Eq. [10], to 1, if the sum is over all diagrams C_i except to vacuum diagrams, while Eq. [6] reduces to the same statement for W in place of the sum of C_i. That does the job! We are led to the conclusion that $W(J,\lambda)$ constitutes indeed the set of connected diagrams, excluding vacuum ones. In other words, the knowledge of the connected parts is enough to characterize *all* diagrams, connected or disconnected. For this reason, it is enough to calculate W instead of Z. This statement, which is not trivial and even counterintuitive at first, receives the following name: "logarithm property of the generating functional" (Z is the functional and the statement can be expressed as $\ln Z = W$). In any case, this property constitutes an extremely important theoretical result in Quantum Field Theory.

Conclusion

When the prominent physicist Richard Feynman invented his famous diagrams, he made the study of particle propagations and scattering processes much more accessible. In a manner of speaking, a kid could understand the meaning of the lines and vertices displayed in a simple Feynman diagram. A kid who knows a surprising amount of mathematics could use the diagram to calculate probability amplitudes and make predictions in a very straightforward way. In fact, field theorist see Feynman diagrams as the most efficient tools available to them for their computational work. In this paper, we have derived the basic results needed for a basic understanding of Feynman diagrams and their useful applications.

References

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