Solving the Geodesic Equation

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Abstract

We find the general form of the geodesic equation and discuss the closed form relation to find Christoffel symbols. We then show how to use metric independence to find Killing vector fields, which allow us to solve the geodesic equation when there are helpful symmetries. We also discuss a more general way to find Killing vector fields, and some of their properties as a Lie algebra.

1 The Variational Method

We will exploit the following variational principle to characterize motion in general relativity:

> The world line of a free test particle between two timelike separated points extremizes the proper time between them.

where a *test particle* is one that is not a significant source of spacetime curvature, and a free particles is one that is only under the influence of curved spacetime. Similarly to classical Lagrangian mechanics, we can use this to deduce the equations of motion for a metric.

The proper time along a timeline worldline between point A and point B for the metric $g_{\mu\nu}$ is given by

$$\tau_{AB} = \int_A^B d\tau = \int_A^B (-g_{\mu\nu}(x)dx^\mu dx^\nu)^{1/2} \tag{1}$$

using the Einstein summation notation, and $\mu, \nu = 0, 1, 2, 3$. We can parameterize the four coordinates with the parameter σ where $\sigma = 0$ at A and $\sigma = 1$ at B. This gives us the following equation for the proper time:

$$\tau_{AB} = \int_0^1 d\sigma \left(-g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \right)^{1/2} \tag{2}$$

We can treat the integrand as a Lagrangian,

$$\mathscr{L} = \left(-g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \right)^{1/2} \tag{3}$$

and it's clear that the world lines extremizing proper time are those that satisfy the Euler-Lagrange equation:

$$\frac{\partial \mathscr{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{\partial \mathscr{L}}{\partial (dx^\mu/d\sigma)} \right) = 0 \tag{4}$$

These four equations together give the equation for the worldline extremizing the proper time. This worldline is called the *geodesic*.

Since the Lagrangian necessarily involves a square root of a summation of terms, taking its derivative will result in a pervasive factor of $1/\mathscr{L}$. However, since $\mathscr{L} = d\tau/d\sigma \implies 1/\mathscr{L} = d\sigma/d\tau$, we can use this to change derivatives with respect to σ to those with respect to τ. Additionally, the second term in (4) implies that the resulting equations will all have second derivatives with respect to σ that can be changed. Thus, it's easy to see that each of the equations will have the form

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \tag{5}$$

where the coefficients $\Gamma^\mu_{\alpha\beta}$ are called the *Christoffel symbols*, which depend on the metric and are taken to be symmetric in the lower indices. These equations together are the *geodesic equation*.

To find the general form for the Christoffel symbols, we first write out the general Euler-Lagrange equation:

$$\frac{1}{2}\left(-g_{\gamma\delta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\delta}{\mathrm{d}\sigma}\right)^{-1/2}\frac{\partial g_{\varepsilon\beta}}{\partial x^\alpha}\frac{\partial x^\varepsilon}{\partial\sigma}\frac{\partial x^\beta}{\partial\sigma} \tag{6}$$

$$-\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(-g_{\gamma\delta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\delta}{\mathrm{d}\sigma}\right)^{-1/2} g_{\alpha\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma}\right] = 0 \tag{7}$$

We know that

$$\left(-g_{\gamma\delta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\delta}{\mathrm{d}\sigma}\right)^{-1/2} = \frac{1}{\mathscr{L}} = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} \tag{8}$$

which, after multiplying by $d\sigma/d\tau$, allows us to simplify (6) to

$$\frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial^\alpha}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\alpha\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right] = 0 \tag{9}$$

Now,

$$\frac{\mathrm{d}g_{\alpha\beta}}{\mathrm{d}\tau} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} \tag{10}$$

allowing us to write (9) as

$$g_{\alpha\beta}\frac{\mathrm{d}^2x^\beta}{\mathrm{d}\tau^2} + \left(-\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0 \tag{11}$$

This can also be written as

$$g_{\alpha\delta}\frac{\mathrm{d}^2x^\delta}{\mathrm{d}\tau^2} = -\frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0 \tag{12}$$

Note that we changed the summation index β to δ for clarity. The first two terms in the parentheses contribute the same; just by changing summation indices, clearly,

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \tag{13}$$

When the geodesic equation is written in the form of (5), we can identify the Christoffel symbols by multiplying that equation by $g_{\alpha\delta}$:

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right) \tag{14}$$

which is the general relation for the Christoffel symbols. This equation can be useful if the metric is diagonal in the coordinate system being used, as then the left hand side only contains a single term; otherwise, we would need to compute the metric inverse $g^{\alpha\delta}$ (where $g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$), a non-trivial task for non-diagonal metrics.

2 Symmetries and Conservation Laws

The task we have given ourselves is, in general, rather intractable. Solving a set of four coupled, second-order ordinary differential equations can be easy for simple metrics, but quickly becomes very difficult for more interesting cases.

To simplify our task, we need to find *conservation laws*, equations that give us first integrals of the equations of motion for free. One that is true for all metrics is the magnitude of the four-velocity:

$$\boldsymbol{u}\cdot\boldsymbol{u} = g_{\alpha\beta}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tau} = -1 \tag{15}$$

Sadly, this is the only universally true conservation law.

However, all is not lost! According to Noether's first theorem, every differentiable symmetry of the action of a system has a corresponding conservation law. In this case, the action is just the proper time, and thus we're looking for symmetries in the metric.

These are not always easy to find, but the simplest symmetries can be found by observing if the metric is independent of any of its coordinates. Then an infinitesimal transformation of that coordinate, $x^{\mu} \to x^{\mu} + \delta x^{\mu}$, leaves the metric unchanged.

We can define a vector field for each symmetry such that, at every point, a vector points along the direction in which the metric doesn't change due to that symmetry. This is called a *Killing vector field*, after the German mathematician Wilhelm Killing. For example, if we have a metric independent of x^1, the Killing field associated with that symmetry is

$$\xi^{\alpha} = (0, 1, 0, 0) \tag{16}$$

We may use the term Killing field and Killing vector interchangeably.

A symmetry implies that there is a conserved quantity along a geodesic. This can be seen by looking at the Euler-Lagrange equation, from which the geodesic equation is derived. If the metric is independent of a coordinate, which without loss of generality we'll say is x^1, then $\partial\mathscr{L}/\partial x^1 = 0$. So, the Euler-Lagrange equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial\mathscr{L}}{\partial(dx^1/d\sigma)}\right) = 0 \tag{17}$$

This means that the quantity inside the derivative is constant along the geodesic. Now,

$$\frac{\partial \mathscr{L}}{\partial(dx^1/d\sigma)} = -g_{1\beta}\frac{1}{\mathscr{L}}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma} \tag{18}$$

$$= -g_{1\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \tag{19}$$

$$= -g_{\alpha\beta}\xi^\alpha u^\beta \tag{20}$$

$$= -\boldsymbol{\xi}\cdot\boldsymbol{u} \tag{21}$$

where ξ^α is a Killing vector and u^β is a four-velocity. Thus, $\boldsymbol{\xi}\cdot\boldsymbol{u}$ is a conserved quantity. We can exploit this to solve geodesic equations.

3 Example: The Plane

The procedure for solving the geodesic equations is best illustrated with a fairly simple example: finding the geodesics on a plane, using polar coordinates to grant a little bit of complexity.

First, the metric for the plane in polar coordinates is

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{22}$$

Then the distance along a curve between A and B is given by

$$S = \int_A^B ds = \int_A^B \sqrt{dr^2 + r^2 d\phi^2} \tag{23}$$

As above, we'll choose a parameter $\sigma \in [0, 1]$. Then,

$$S = \int_0^1 d\sigma \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\sigma}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2} \tag{24}$$

Taking the Lagrangian as the integrand and plugging it into the Euler-Lagrange equations for r and ϕ, we have

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{\mathscr{L}}\frac{\mathrm{d}r}{\mathrm{d}\sigma}\right) = \frac{r}{\mathscr{L}}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right) \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{r^2}{\mathscr{L}}\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2 = 0 \tag{26}$$

Now, using the fact that $\mathscr{L} = ds/d\sigma$, we have

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = r\left(\frac{\mathrm{d}\phi}{\mathrm{d}}\right)^2 \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}\right) = 0 \quad \to \quad \frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} = -\frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} \tag{28}$$

Clearly, the only non-zero Christoffel symbols are

$$\Gamma^r_{\phi\phi} = -r \tag{29}$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \tag{30}$$

To begin, we can divide (22) by ds to get the first integral, corresponding to $\boldsymbol{u} \cdot \boldsymbol{u} = 1$:

$$1 = \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 \tag{31}$$

Since the metric is independent of ϕ, we have the Killing vector $\boldsymbol{\xi} = (0, 1)$. So we have the conserved quantity

$$\boldsymbol{\xi} \cdot \boldsymbol{u} = g_{ij} u^i u^j = r^2 \frac{\mathrm{d}\phi}{\mathrm{d}s} \equiv \ell \tag{32}$$

Using this in (31), we get

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \left(1 - \frac{\ell^2}{r^2}\right)^{1/2} \tag{33}$$

Dividing (32) by (33), we get

$$\frac{d\phi/ds}{dr/ds} = \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\ell}{r^2}\left(1 - \frac{\ell^2}{r^2}\right)^{1/2} \tag{34}$$

We can integrate this with respect to r to get

$$\phi = \arccos\left(\frac{\ell}{r}\right) + \phi_* \quad \rightarrow \quad r\cos(\phi - \phi_*) = \ell \tag{35}$$

where ϕ_* is an integration constant. Using a trigonometric identity to expand the cosine and the fact that $x \equiv r\cos(\phi)$ and $y \equiv r\sin(\phi)$, we have

$$x\cos(\phi_*) + y\sin(\phi_*) = \ell \tag{36}$$

This is just an equation for a straight line! Thus, the solution to the geodesic equation comes out to what we would expect.

4 More Killing Vectors

We previously discussed the easiest method of finding Killing vectors: read off the coordinates of which the metric is independent. However, this will usually not give us all of the symmetries (or even *any* of them). Take, for example, the 3D Cartesian metric,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{37}$$

This obviously has three Killing vectors:

$$\boldsymbol{\xi}_1 = (1, 0, 0) \tag{38}$$
$$\boldsymbol{\xi}_2 = (0, 1, 0) \tag{39}$$
$$\boldsymbol{\xi}_3 = (0, 0, 1) \tag{40}$$

However, consider the spherical metric, which describes the same space:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{41}$$

This metric is independent of ϕ, so in spherical coordinates, it should have the Killing vector $\boldsymbol{\xi}_4 = (0, 0, 1)$, or in Cartesian, $(-y, x, 0)$. In fact, 3D Euclidean space has 6 Killing vectors: three translations and three rotations. The metric we started with didn't make these rotational symmetries obvious.

To find all of the Killing vectors, we'll start with the requirement that the metric be unchanged under a constant coordinate transformation $x \to x'$:

$$g_{\rho\sigma}(x') = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \tag{42}$$

Does this equation have any solutions?

In general, this is a set of equations that are difficult to solve, but looking at an infinitesimal case makes it considerably easier. Take x' and x to be related by $x'^\mu = x^\mu + \varepsilon\xi^\mu(x)$, where $\varepsilon << 1$, and ξ^μ is some vector (with suggestive notation). Now, we'll expand (42) out to order ε. Setting

$$\frac{\partial x^\mu}{\partial x'^\rho} = \delta^\mu_\rho - \varepsilon\partial_\rho\xi^\mu(x) + O(\varepsilon^2) \tag{43}$$

and plugging into (42), the right hand side becomes

$$\begin{aligned} g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} &= g_{\mu\nu}(\delta^\mu_\rho - \varepsilon\partial_\rho\xi^\mu)(\delta^\nu_\sigma - \varepsilon\partial_\sigma\xi^\nu) && (44)\\ &= (g_{\rho\nu} - g_{\mu\nu}\varepsilon\partial_\rho\xi^\mu)(\delta^\nu_\sigma - \varepsilon\partial_\sigma\xi^\nu) && (45)\\ &= g_{\rho\sigma} - g_{\rho\nu}\varepsilon\partial_\sigma\xi^\nu - g_{\mu\sigma}\varepsilon\partial_\rho\xi^\mu + O(\varepsilon^2) && (46)\end{aligned}$$

We Taylor expand the left hand side, giving

$$g_{\rho\sigma}(x^\mu + \varepsilon\xi^\mu) = g_{\rho\sigma}(x) + \varepsilon\xi^\lambda\partial_\lambda g_{\rho\sigma} + O(\varepsilon^2) \tag{47}$$

Neglecting terms of order ε^2 and putting the left and right sides back together, we have

$$\begin{aligned} g_{\rho\sigma}(x) + \varepsilon\xi^\lambda\partial_\lambda g_{\rho\sigma} &= g_{\rho\sigma} - g_{\rho\nu}\varepsilon\partial_\sigma\xi^\nu - g_{\mu\sigma}\varepsilon\partial_\rho\xi^\mu && (48)\\ \to \quad g_{\mu\sigma}\partial_\rho\xi^\mu + g_{\rho\nu}\partial_\sigma\xi^\nu + \xi^\lambda\partial_\lambda g_{\rho\sigma} &= 0 && (49)\end{aligned}$$

Since $g_{\rho\sigma}$ is symmetric, this is actually 10 equations; thus, there are at maximum (for a 4-dimensional spacetime) 10 symmetries. Or, more generally, $\frac{1}{2}D(D+1)$ for dimension D. A metric with all of those symmetries is said to be *maximally symmetric*.

Let's test this out on the very simple 3D Cartesian metric. We obtain from (49) the following six equations:

$$\begin{aligned} \partial_x\xi^x &= 0 && (50)\\ \partial_x\xi^y + \partial_y\xi^x &= 0 && (51)\\ \partial_x\xi^z + \partial_z\xi^x &= 0 && (52)\\ \partial_y\xi^y &= 0 && (53)\\ \partial_z\xi^y + \partial_y\xi^z &= 0 && (54)\\ \partial_z\xi^z &= 0 && (55)\end{aligned}$$

These are fairly easily solved. For example, act ∂_x on the (52) and use (51), letting us obtain $\partial_x^2\xi^y = 0$, etc. We end up finding the six Killing vectors we would expect:

$$\begin{array}{lll} \boldsymbol{\xi}_1 = (1,0,0) & \boldsymbol{\xi}_2 = (0,1,0) & \boldsymbol{\xi}_3 = (0,0,1) \\ \boldsymbol{\xi}_4 = (-y,x,0) & \boldsymbol{\xi}_5 = (0,-z,y) & \boldsymbol{\xi}_6 = (z,0,-x) \end{array}$$

Now, we still need to show that these Killing vectors lead to conserved quantities, since our previous proof relied on the metric being independent of a coordinate. This will only be a proof sketch, as all of the background is out of the scope of this paper.

Consider a geodesic $X^\mu(\tau)$, with the tangent velocity vector $V^\mu(\tau) = dX^\mu/d\tau$. Let $\xi(x)$ be a Killing vector field of the metric describing this spacetime. We wish to show that $\xi_\mu(X^\mu)V^\mu$ is conserved along the geodesic.

We first define the covariant derivative:

$$D_\lambda W^\mu = \partial_\lambda W^\mu + \Gamma^\mu_{\lambda\nu} W^\nu \tag{56}$$

This is basically because the standard derivative doesn't transform properly, while this definition does, so it's more useful for our purposes. It works in a very similar way.

Now, act the covariant derivative on the quantity:

$$V^\nu D_\nu(\xi_\mu V^\mu) = V^\nu V^\mu D_\nu \xi_\mu + \xi_\mu(V^\nu D_\nu V^\mu) \tag{57}$$

by the product rule. Now, the second term is actually

$$V^\nu D_\nu V^\mu = \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\right) = 0 \tag{58}$$

according to the geodesic equation. So now we have just $V^\nu V^\mu D_\nu \xi_\mu$. Notice that we can rewrite (49) using the covariant derivative and the formula for the Christoffel symbols as

$$D_\rho \xi_\sigma + D_\sigma \xi_\rho = 0 \tag{59}$$

This implies that $D_\nu \xi_\mu$ is antisymmetric in its indices. Thus, the whole term is equal to zero. Since the (covariant) derivative of $\xi_\mu(X^\mu)V^\mu$ is 0 along a geodesic, then it must be conserved along that geodesic.

5 Killing Vector Lie Properties

We will now briefly describe some of the Lie algebraic properties of Killing vectors. This will be mostly without proof, as this is a little out of the scope of this course.

We can always associate a vector V^μ with a differential operator $V^\mu \partial_\mu$. Thus, we can write Killing vectors as a summation of functions multiplied by differential operators. For example, we can write $\boldsymbol{\xi}_4$ from the above formulation as

$$\xi_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{60}$$

Interestingly, if we commute, say, $\boldsymbol{\xi}_4$ and $\boldsymbol{\xi}_5$, we find that

$$[\xi_4, \xi_5] = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right) - \left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \tag{61}$$

$$= \xi_6 \tag{62}$$

In fact, it can be shown that, for any three Killing vectors for the same metric,

$$[\xi_a, \xi_b] = \varepsilon_{abc}\xi_c \tag{63}$$

where ε_{abc} is the permutation symbol, defined such that, if $\varepsilon_{ijk} = 1$, any flipping of indices flips its sign.

Now, if we begin with a Lie group G and a subgroup H, and say that g_1 is equivalent to g_2 for $g_1, g_2 \in G$ if there exists a subgroup element h such that $g_1 = g_2 h$. Then we say that g_1 and g_2 belong to the same equivalence class if they are equivalent. Finally, we can define a manifold by associating each equivalence class with a point; this is called a *coset manifold* G/H.

On a coset manifold G/H, the Killing vectors actually satisfy

$$[\xi_a, \xi_b] = f_{abc}\xi_c \tag{64}$$

where f_{abc} are the structure constants of the Lie algebra of the group G.

6 Conclusion

In this paper, we discussed the origin of the geodesic equation and the method of solving it, namely that of finding symmetries and using Killing vectors.

We are very lucky that many physically interesting situations have these nice symmetry properties. When these don't exist for a metric, we have to resort to numerical simulations, which in Einstein's day would have been either very difficult or impossible. Using these symmetries, we can actually study many interesting metrics analytically, including the famed Schwarzchild metric for space near a massive body. Clearly, symmetries in physics are very important to our understanding and ability to understand complicated physical situations, and methods of finding new ones are critical to progress.

References

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