Gravitational Waves

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Abstract

This paper is based on my lecture for the Kapitza Society. In it, we discuss gravitational waves, with a focus on linearized gravitational waves. We address the mathematical representation of linearized gravitational waves, discuss their detection, and construct their most general form. We conclude with a discussion of modern efforts to detect them.

1 Introduction

Einstein's Theory of General Relativity explains that mass produces a curvature of spacetime. This has been discussed at length in this course through the study of the geometries that arise due to different astronomical phenomena. General Relativity also predicts that ripples of spacetime, that propagate at the speed of light, are generated by masses in non-spherical, nonuniform motion. These ripples are called *gravitational waves*, and have since been detected on Earth. We will discuss *linearized gravitational waves*: weak gravitational waves propagating in a nearly flat spacetime, entirely devoid of matter.

2 Linearized Gravitational Waves

The most simple example of a gravitational wave is a *plane wave*. Such waves propagate in one direction, known as *longitudinal*, and independent of the other two perpendicular directions, which are known as *transverse*. The wave is the same in both transverse directions, hence the name *plane wave*.

As we already know, in the (t, x, y, z) coordinates of an inertial frame, the metric of a flat spacetime is $g_{\alpha\beta} = \eta_{\alpha\beta}$, with $\eta_{\alpha\beta} = diag(-1, 1, 1, 1)$. As such, for a metric that is *close* to flat, we can write:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}(x) \tag{1}$$

Where the amplitudes $h_{\alpha\beta}(x)$ represent small perturbations to this flat spacetime metric, and are accordingly called *metric perturbations*. For a general plane gravitational wave propagating in the z-direction, we have:

$$h_{\alpha\beta}(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} f(t - z)$$

Where $f(t-z)$ is any function of $t-z$ where $|f(t-z)| \ll 1$. With such a perturbation, the line element for the spacetime is given by:

$$ds^2 = -dt^2 + [1 + f(t-z)]dx^2 + [1 - f(t-z)]dy^2 + dz^2 \quad (2)$$

The geometry in (2) represents a ripple of curvature propagating in the positive z-direction with speed 1 (the speed of light). The amplitude and shape of the ripple, as one would expect, are determined by $f(t-z)$. As both the various $h_{\alpha\beta}$ and $f(t-z)$ are dimensionless, the amplitude of a gravitational wave is also dimensionless.

For example, if one were to choose $f(t-z) = a \exp[\frac{-(t-z)^2}{\sigma^2}]$, the ripple would be a Gaussian wave packet with width σ and height a that would propagate along the z-axis at the speed of light without changing shape.

It is, however, important to note that the metric in (2) does not solve the Einstein Equation exactly. It does, however solve the equation when it is expanded to the first order (linearized) in the amplitude of the wave. Linearized waves are especially useful for approximations of gravitational waves with small amplitudes, and also can be added to one another to produce other linearized gravitational waves that solve Einstein's Equation to the same degree of accuracy, something that cannot be done with solutions of the full equation.

An important result that we did not get to cover in this course is that any linearized gravitational wave can be represented as the sum of one waves of the form in (2) and another wave of a similar form with different polarizations.

3 Detection of Gravitational Waves

When we have discussed other spacetimes, we have used the motion of test bodies to discuss the curvature of spacetime, but talking about the motion of a single test body will not be sufficient in the discussion of gravitational waves. For example, if we had our test body that is at rest with respect to some frame, it will remain at rest there whether a gravitational wave is present or not. As such, as we move forward, we will work with the relative motion of two or more test bodies when discussing gravitational waves.

We will now discuss how two test bodies would be affected by the gravitational wave packed in (2). Take two test bodies A and B, at $(0,0,0)$ and (x_B, y_B, z_B) respectively. The initial four-velocities are given by:

$$u^{\alpha}_{(A)} = u^{\alpha}_{(B)} = (1, \vec{0}) \quad (3)$$

Before the gravitational wave passes, the spacetime is flat and we have:

$$x^i_{(A)}(\tau) = (0,0,0); x^i_{(B)}(\tau) = (x_B, y_B, z_B) \quad (4)$$

The geodesic equation must be solved for each particle using the metric (2) in order to predict the motion of the particles after the wave packet affects them. Because the amplitude of the wave is small, we will solve for first order corrections $\delta x^i_{(A)}(\tau)$ and $\delta x^i_{(B)}(\tau)$. The geodesic equation for the spacial coordinates, $x^i(\tau)$ of either particle is given by:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \quad (5)$$

These equations simplify due to only needing to calculate first order changes. $\Gamma^i_{\alpha\beta}$ vanishes in the unperturbed, flat spacetime, so:

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{\alpha\beta}u^\alpha u^\beta = -\delta\Gamma^i_{tt} \tag{6}$$

Where $\delta\Gamma^i_{\alpha\beta}$ are the first order changes in the Γ's and the u^α are the unperturbed four-velocities given by (3). The Christoffel symbol Γ^i_{tt} vanishes for our metric (2), so $\delta\Gamma^i_{tt}$ does as well. Thus, we have:

$$\frac{d^2\delta x^i}{d\tau^2} = 0 \tag{7}$$

Initially we have that $\delta x^i = 0$ and that the test masses are at rest, so $\frac{d\delta x^i}{d\tau} = 0$ for all τ and both test masses:

$$\delta x^i_{(A)} = \delta x^i_{(B)} = 0 \tag{8}$$

The result of this is that there is no change in the coordinate positions of either of the test particles as the wave passes (to the first order in amplitude of the wave). The distance between the masses, however, does change with time, as will be demonstrated by the example that follows.

3.1 Example: The Change in Distance Between Two Test Masses

We will consider a wave of the form (2) traveling in the z-direction and two test masses, one at the origin, and one at position $(L_*, 0, 0)$ in order to calculate the change $\delta L(t)$ in the distance between them. The distance between the test masses will be L_* in the unperturbed spacetime. In the spacetime given in (2), the distance between them (as measured along the x-axis, $L(t)$ and $\delta L(t)$ are given by:

$$\int_0^{L_*} dx[1 + h_{xx}(t, 0)]^{1/2} \approx L_*[1 + \frac{1}{2}h_{xx}(t, 0)] \tag{9}$$

$$\frac{\delta L(t)}{L_*} = \frac{1}{2}h_{xx}(t, 0) \tag{10}$$

As such, the distance between the test masses changes with time according to the variation of the wave. If the wave were to have a definite frequency ω, amplitude a, and phase δ so that $f(t - z) = a\sin(\omega t + \delta)$, one would find that:

$$\frac{\delta L(t)}{L_*} = \frac{1}{2}a\sin(\omega t + \delta) \tag{11}$$

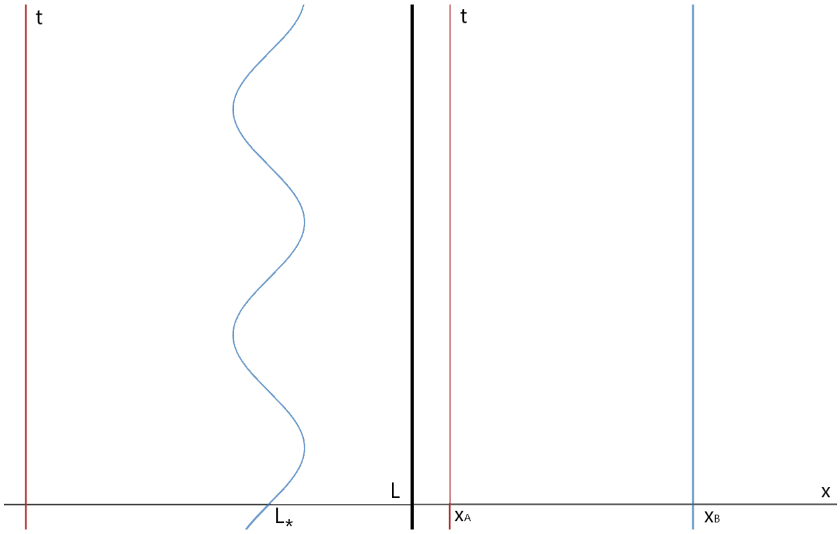

Figure 1: The motion of test particles in the gravitational wave spacetime. The two test particles are initially at $x = 0$ and $x = L_*$. As the wave passes, the coordinate distance does not change, but the distance between them, $L(t)$, does in accordance with the oscillations of the wave.

3.2 Generalization of Results

The work in that example can be generalized to the case where the first test particle is at the origin and the second one is at an arbitrary point in the plane transverse to the direction of the wave's propagation. Let the second test mass be a distance L_* from the origin along the direction of a unit vector $\vec{n}$ in the plane $z = 0$. The ratio $\frac{L}{L_*}$ is called the fractional strain produced by the gravitational wave, and it and the distance, $L(t)$, between the two test charges (as calculated along the path that is a straight line in the unperturbed spacetime), will then be given by:

$$\frac{\delta L(t)}{L_*} = \frac{1}{2} h_{ij}(t, 0) n^i n^j \qquad (12)$$

Despite the fact that the path we calculated the path along was straight in the flat spacetime and the fact that the coordinate distance was unchanged by the gravitational wave, the path will not be a geodesic in the curved spacetime of the wave. For the sake of laser interferometry, it is more viable to calculate the separation of particles along the path of a light ray between the test masses. In flat spacetime, light rays follow straight line paths, as they do in any spacetime, but straight paths are paths of extremal distance in flat spacetime, from which there are only second order changes (with regard to distance), so (12) also gives the change in distance along the path of a light ray (first order in amplitude).

4 Gravitational Wave Polarization

The gravitational wave metric (2) causes no change to the distance between masses on the z-axis (longitudinal direction). The perturbation h_{zz} also vanishes from the generalized version of (9). This means that only transverse separations of test particles change with time as the wave packet passes. As such, gravitational waves are transverse.

We will now discuss the effects of a passing gravitational wave on test masses arranged in a circle in the $x - y$ plane at $z = 0$ with another mass at the center. We will have a gravitational wave of the form in (2) with $f(t - z) = a\sin(\omega(t - z))$. As we now know, the coordinate positions of these test charges will be unchanged, but the distance between each test mass and the one at the center will change with time. To calculate these distances, we will introduce new coordinates (X, Y) for the $x - y$ plane at $z = 0$:

$$X = (1 + \frac{1}{2}a\sin(\omega t))x; Y = (1 + \frac{1}{2}a\sin(\omega t))y \tag{13}$$

This clever choice of coordinates has the line element $dS^2 = dX^2 + dY^2$, which is the same as that of a flat Euclidean plane (ignoring corrections of order a^2 that will be negligible for waves with small amplitude). The distances between the test masses in the $x - y$ plane can now be calculated from their $X(t)$ and $Y(t)$ components (to the first order with respect to the amplitude of the wave).

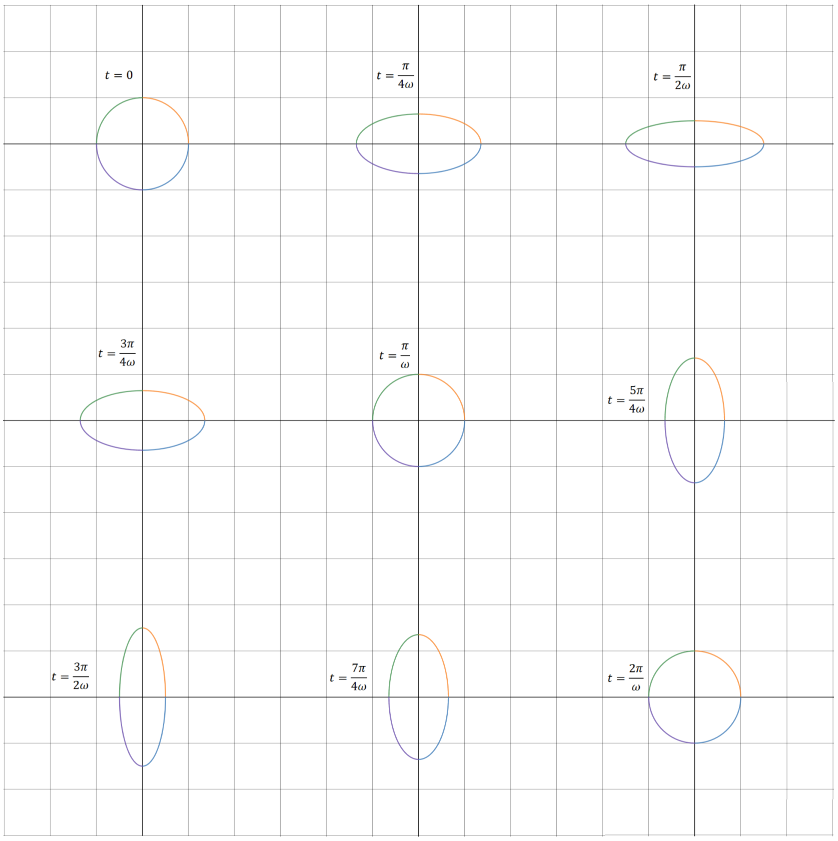

Figure 2: Shows the time progression of a ring of test masses in the plane transverse to the propagation of the wave in (13). The grid lines are a (1 amplitude) apart, and ω is the frequency from (13. The X and Y directions are expanding and contracting out of phase with one another ($\frac{\pi}{4}$ radians apart).

As was mentioned before, the metric in (2) is not the most general one, but is instead only accounting for a single *polarization* out of the two possible independent ones that we can have.

To find another polarization, we can rotate the x and y axes by some angle θ. We can choose an angle (we will use 45^o and find the relation between (x, y) and our new, rotated coordinates (x', y'):

$$x = \frac{1}{\sqrt{2}}(x' + y'); y = \frac{1}{\sqrt{2}}(x' - y') \tag{14}$$

By substituting these into (2), we can see how our metric changes. The flat spacetime, $\eta_{\alpha\beta}$, remains unchanged by rotations, and our new coordinates are not physically different from the old ones, so we will have a new solution to our linearized Einstein equation. These facts give:

$$h_{x'x'} = 0; h_{x'y'} = h_{y'x'} = h_{xx} = -h_{yy}; h_{y'y'} = 0 \tag{15}$$

$$h_{\alpha\beta}(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} f(t - z) \tag{16}$$

It is easily see that this solution and our original solution will be linearly independent. The behavior of test particles will also be the same, but rotated by 45^o. Gravitational waves of the form in (2) are usually denoted to have a + polarization, while those of the form in (16) are denoted to have a x polarization. We can take a linear combination of two arbitrary gravitational waves of these forms, and thus obtain the general form for a linearized gravitational wave propagating in the z-direction:

$$h_{\alpha\beta}(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_x(t-z) & 0 \\ 0 & f_x(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

5 Gravitational Wave Interferometers

As was stated in the Introduction, gravitational waves have been detected on earth. We will discuss how similar detectors to the ones that were used function.

5.1 Theory

The detectors that are used at the The Laser Interferometer Gravitational-Wave Observatory (LIGO), the physics experiment/observatory that detected gravitational waves, are, as the name suggests, laser interferometers. Specifically, LIGO makes use of Michaelson Interferometers

Laser interferometers utilize the constructive and destructive interference of beams of light to measure changes in distance to a high degree of precision. In Michaelson Interferometers, shown in Figure (3) below, a laser emits a beam of light of a single wavelength that then enters a beam splitter where it is split into two beams that then travel down two perpendicular paths. At the end of each path, there is a mirror that reflects the beams back toward the beam splitter. Before reaching the beam splitter, however, there is a partially reflective mirror. The photons then are either reflected back toward the mirrors or pass through the beam splitter, where some would then go to the detector where these two beams (now not necessarily in phase) will interfere. By knowing the initial lengths of the paths from the beam splitter to the two mirrors ($L_{(x)}$ and $L_{(y)}$), the wavelength of light from the beams (λ, and the interference pattern that is observed, one can calculate the difference between the lengths of the two arms of the interferometer:

$$\Delta L \equiv L_{(x)} - L(y) = n\lambda; (n = 0, 1, 2, 3, ...); \text{for constructive interference} \tag{18}$$

$$\Delta L \equiv L_{(x)} - L(y) = (n + \frac{1}{2})\lambda; (n = 0, 1, 2, 3, ...); \text{for destructive interference} \tag{19}$$

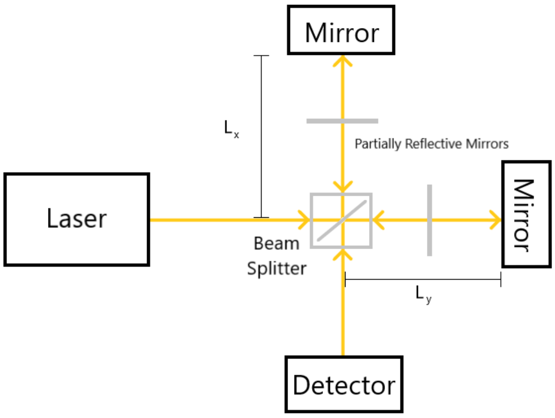

Figure 3: Shows the layout of a typical Michaelson Interferometer.

For the purposes of measuring gravitational waves, the beam splitter and mirrors will be attached to test masses that are hung and allowed to swing freely in horizontal directions. This setup allows us to measure the change in distance between the beam splitter test mass and each of the mirror masses in terms of half wavelengths of the light we are using. The partially reflective mirrors (when included) act to extend the length of the interferometer's arms by also being attached to hanging test masses, but the general principle is the same.
The difference in arm lengths, $\Delta L \equiv L_{(x)} - L(y)$, will change with regard to time as a gravitational wave acts on the system. As ΔL changes, the interference will go between being constructive and destructive, and the plot of the intensity of the light hitting the detector will also vary with regard to time. One can think of the beam splitter mass as the central mass in each of the pictures in Figure (2), and each of the mirrors as one of the other masses in the same picture. If we assume that the gravitational wave is of the form in (2) with a definite frequency (ω), as the wave passes, $L_{(x)}$ and $L(y)$ will expand and contract out of phase with each other. Moreover, if we assume that the interferometer arms are oriented along the x- and y-axes, we get that:

$$\frac{\delta L_{(x)}}{L_{(x)}} = +\frac{1}{2} a \sin(\omega t); \frac{\delta L_{(y)}}{L_{(y)}} = -\frac{1}{2} a \sin(\omega t) \tag{20}$$

The amplitude and frequency can be determined by looking at the interference pattern that would be picked up by the detector. For a gravitational wave with a given amplitude, longer interferometer arms will allow for a more sensitive detector. It is worth noting that actual detectors, like LIGO, are more sophisticated than these Michaelson interferometers, but they are based on them and operate on the same ideas.

5.2 Detection

The Laser Interferometer Gravitational-Wave Observatory was the first observatory to detect gravitational waves. On September 14, 2015, less than two days after they turned on the detectors after being upgraded to have a strain ($\frac{\delta L}{L}$) sensitivity of of 10^{-23} [2], the first ever detection of gravitational waves on Earth occurred. The observatory has interferometer arms 4 km in length, and the ability to pick up waves with frequencies ranging from 10-10kHz. The next observation run of LIGO is scheduled for February 2019.

References

[1] Hartle, J. *Gravity: An Introduction to Einstein's General Relativity (4th ed.)*. Pearson.

[2] Martynov, D., Hall, E., Abbott, B., Abbott, R., Abbott, T., & Adams, C. et al. (2016). Sensitivity of the Advanced LIGO detectors at the beginning of gravitational wave astronomy. *Physical Review* D, 93(11). doi:10.1103/physrevd.93.112004