Modeling Particle Interactions with Feynman Diagrams

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Abstract

We build the basic theory of Feynman diagrams, motivated by understanding particle interactions. Following Wick and Schwinger, we solve a simple problem of two mesons interacting and resulting in two mesons again, and demonstrate how Feynman diagrams can be used to solve the problem much more easily. We then explore how they can be used to quickly find the amplitude of a two particle interaction that yields four particles, and then show the breakdown of the theory when the diagrams contain loops (and motivate renormalization). We also show how vacuum fluctuations arise. All information is based on, and all figures are drawn from, *Quantum Field Theory in a Nutshell* by A. Zee [1].

Free field theory is fairly easy to solve just using path integrals, since the defining path integral is just a Gaussian, and we have standard rules to work those out. But this approximation doesn't allow for interactions, because it is fundamentally linear: just in the same way that solutions to a harmonic oscillator are waves that can travel past each other without changing each other in the slightest, free field theory doesn't allow us to model interactions because the particles just fly by. We need to include some other terms in the Lagrangian to model these more complicated situations. We'll add a quadratic term, $\frac{\lambda}{4!}\varphi^4$, and see what impact it has on our model.

Consider the setup in figure 1. This is not a Feynman diagram, only a representation of the general situation we are trying to model. The numbered circles represent sources and sinks (depending on whether they have

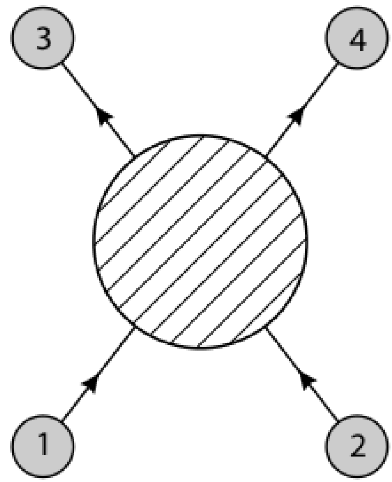

Figure 1

an outgoing or incoming arrow), and the big central circle represents something happening between the creation (or entrance) of our particles and their destruction (or exit). Our goal is to work out what's going on in that middle bit, in the situation where all our particles are mesons.

We should be able to do this by evaluating the following path integral:

$$Z(J) = \int D\varphi \; e^{i\int d^4x\left[\frac{1}{2}\left((\partial\varphi)^2 - m^2\varphi^2\right) - \frac{\lambda}{4!}\varphi^4 + J\varphi\right]} \tag{1}$$

which is very similar to our free field theory, but with the added $\frac{\lambda}{4!}\varphi^4$ term so as to make it anharmonic. Without this term, we have a situation very similar to a harmonic oscillator, and the particles will simply pass right through each other. As usual, the φ is our field variable, and J is a function representing our sources and sinks. Since we're using the setup in figure 1, we specifically need to find the term containing $J(x_1)J(x_2)J(x_3)J(x_4)$. This is just the four-point Green's function, $G(x_1, x_2, x_3, x_4)$.

We can go about evaluating this integral in two ways. We will begin by following Wick's method. As a reminder, our first evaluation step is

$$
\begin{aligned}
Z(J) &= Z(0,0)\sum_{s=1}^{\infty}\frac{i^s}{s!}\quad\int dx_1\dots dx_s\; J(x_1)\dots J(x_s)G^{(s)}(x_1,\dots,x_s)\\
&= Z(0,0)\sum_{s=1}^{\infty}\frac{i^s}{s!}\quad\int dx_1\dots dx_s\; J(x_1)\dots J(x_s)\\
&\qquad\qquad\qquad\times\int D\varphi\; e^{i\int d^4x\,\left[\frac{1}{2}\left((\partial\varphi)^2-m^2\varphi^2\right)-\frac{\lambda}{4!}\varphi^4\right]}\;\varphi(x_1)\dots\varphi(x_s)
\end{aligned}
$$

.

This shows us our four-point Green's function:

$$
\begin{aligned}
G(x_1,x_2,x_3,x_4) &= \frac{1}{Z(0,0)}\int D\varphi\quad e^{i\int d^4x\,\left[\frac{1}{2}\left((\partial\varphi)^2-m^2\varphi^2\right)-\frac{\lambda}{4!}\varphi^4\right]}\;\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\\
&= \frac{1}{Z(0,0)}\int D\varphi\quad e^{i\int d^4x\,\left[\frac{1}{2}\left((\partial\varphi)^2-m^2\varphi^2\right)-\frac{\lambda}{4!}\varphi^4\right]}\;\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\\
&\qquad\qquad\qquad\times\; e^{\frac{i\lambda}{4!}\int d^4w\,\varphi^4}\\
&\approx \frac{1}{Z(0,0)}\int D\varphi\quad e^{i\int d^4x\left[\frac{1}{2}\left((\partial\varphi)^2-m^2\varphi^2\right)-\frac{\lambda}{4!}\varphi^4\right]}\;\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\\
&\qquad\qquad\qquad\times\left(1-\frac{i\lambda}{4!}\int d^4w\;\varphi^4\right).
\end{aligned}
$$

The first term is the standard four-point term, so we really just need to focus on the second term:

$$
\begin{aligned}
&-\frac{1}{Z(0,0)}\frac{i\lambda}{4!}\int d^4w\int D\varphi\\
&\qquad e^{i\int d^4x\left[\frac{1}{2}\left((\partial\varphi)^2-m^2\varphi^2\right)-\frac{\lambda}{4!}\varphi^4\right]}\;\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\varphi^4(w).
\end{aligned}
\tag{2}
$$

Wick contracting this gives us the correction term:

$$
-i\lambda\int d^4w\; D(x_1-w)D(x_2-w)D(x_3-w)D(x_4-w). \tag{3}
$$

As a check, we can also do this calculation the Schwinger way. We begin by splitting (1) into two pieces:

$$
\begin{aligned}
Z(J) &= Z(0,0)e^{-i\frac{\lambda}{4!}\int d^4w\,\left(\frac{\delta}{\delta J(w)}\right)^4}\;e^{-\frac{1}{2}\iint d^4x\,d^4y\;J(x)D(x-y)J(y)}\\
&\approx Z(0,0)\left(-\frac{i}{4}\lambda\right)\int d^4w\;\left(\frac{\delta}{\delta J(w)}\right)^4\left(\frac{i^4}{4!\;2^4}\right)\left[\iint d^4x\;d^4y\;J(x)D(x-y)J(y)\right]^4.
\end{aligned}
\tag{4}
$$

We've dropped some terms here, since the lower-order terms don't have enough J's, and the higher-order ones have too many. Remember, we're looking specifically for the term with four J's; the above expression will result in exactly that, because we begin with eight, and lose four to the derivatives.

We'll introduce some simplifying notation: $J_a = J(x_a)$, $\int_a = \int d^4x_a$, and $D_{ab} = D(x_a - x_b)$ Then dropping overall numerical factors from (4) gives us

$$\sim -i\lambda \int_w \left(\frac{\delta}{\delta J_w}\right)^4 \int_a \int_b \int_c \int_d \int_e \int_f \int_g \int_h D_{ae} D_{bf} D_{cg} D_{dh} J_a J_b J_c J_d J_e J_f J_g J_h. \tag{5}$$

The four $\frac{\delta}{\delta J_w}$'s hit the J's in all possible combinations, but in fact, most of the resulting terms result in diagrams that would be disconnected, and thus not relevant to our physical situation. The remaining term is

$$\sim -i\lambda \int_w \int_a \int_b \int_c \int_d D_{aw} D_{bw} D_{cw} D_{dw} J_a J_b J_c J_d. \tag{6}$$

which comes from the derivatives hitting the J_e, J_f, J_g, and J_h terms, and setting x_e, x_f, x_g, x_h all to w.

We can set $J(x)$ equal to the sum of four delta functions peaked at x_1, x_2, x_3, and x_4, which represents our physical situation. This lets us further evaluate (6) to

$$\sim -i\lambda \int_w D_{1w} D_{2w} D_{3w} D_{4w}.$$

which is the same as equation (3).

How do we interpret this result? Each of the D_{iw} terms stands for a single line in figure 2. These mesons propagate from the points x_1 and x_2 to some point in spacetime marked by w with an amplitude $D(x_1 - w)D(x_2 - w)$, scatter at that point with amplitude $-i\lambda$, and then propagate from w to x_3 and x_4 with amplitude $D(w - x_3)D(w - x_4)$ (note that $D(x) = D(-x)$. Then we integrate w over all of spacetime, meaning that the interaction point could be anywhere, but each point is weighted by the amplitude involving these functions.

This was quite a bit of work to figure out, and this was only a simple example. Can we figure out the rules here, and just use those instead? In

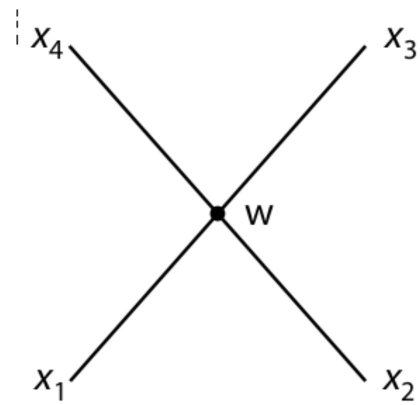

Figure 2

the above example, we could just think about it first in exactly the way just described. Associate $-i\lambda$ with a scattering point, $D(x_1-w)$ with propagation from x_1 to w, and so forth.

Well, since we're talking about it, that might be a bit of a giveaway that it is possible. This is easier to do in momentum space, so we'll switch there first. We have that

$$D(x_a - w) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{\pm ik_a(x_a-w)}}{k_a^2 - m^2 + i\varepsilon}.$$

where we've used a $\pm$ since we have that freedom, and it will make things easier to see in the end.

We can plug this into equation (3) and switch the order of integration. Then with appropriate choice of signs, the inner integral gives

$$\int d^4w \; e^{-i(k_1+k_2-k_3-k_4)} = (2\pi)^4\delta^{(4)}(k_1 + k_2 - k_3 - k_4).$$

So the fact that the interaction can occur anywhere in space actually results in conservation of momentum!

The momentum space Feynman rules are the following:

1. Draw Feynman diagram of the process

2. Label each line with a momentum k and associate it with the propagator $\frac{i}{k^2-m^2+i\varepsilon}$

3. Associate with each interaction vertex the coupling factor $-i\lambda$ and the momentum conservation term $(2\pi)^4\delta^{(4)}\left(\sum_i k_i - \sum_j k_j\right)$

4. Integrate momenta associated with internal lines with the measure $\frac{1}{(2\pi)^4}d^4k$

5. Finally, attach symmetry factors that originate from the various ways the $\frac{\delta}{\delta J}$'s can hit all the J's

Applying these rules to figure (2) yields us the amplitude

$$\prod_{a=1}^{4}\left(\frac{i}{k_a^2 - m^2 + i\varepsilon}\right)(-i\lambda)(2\pi)^4\delta^{(4)}(k_1 + k_2 - k_3 - k_4).$$

Notice that many of these terms will be present in entire classes of diagrams. We don't really need to drag around the $\prod()$ term, since it will appear in any diagram involving two mesons scattering into two mesons. This is actively harmful for external ("real") particles, since they are always "on-shell" (ie, $k_a^2 - m^2 = 0$). Additionally, since there's always going to be an overall momentum conservation factor, we don't need to write down the δ function either. These conditions give us the following additional convenience rules:

6. Don't associate a propagator with the external lines

7. The δ function for overall momentum conservation is understood

Applying these additional rules to figure (2) yields the amplitude $\mathcal{M} = -i\lambda$.

We can now easily describe a wide variety of situations, such as how two colliding mesons can in fact produce four mesons!

A diagram of this situation appears in figure (3). Since there are two vertices, we are looking for a term second-order in λ. We can "amputate the external legs" according to rule 6, and include the propagator associated with the line marked q. We also need momentum conservation δ functions for each of the vertices. Finally, we integrate over the internal momentum q to get

$$\begin{aligned}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m^2 + i\varepsilon}(2\pi)^4\delta^{(4)}(k_1 + k_2 - k_3 - q)(2\pi)^4\delta^{(4)}(q - (k_4 + k_5 + k_6))\\ = (-i\lambda)^2\frac{i}{(k_4 + k_5 + k_6)^2 - m^2 + i\varepsilon}(2\pi)^4\delta^{(4)}(k_1 + k_2 - (k_3 + k_4 + k_5 + k_6)).\end{aligned}$$

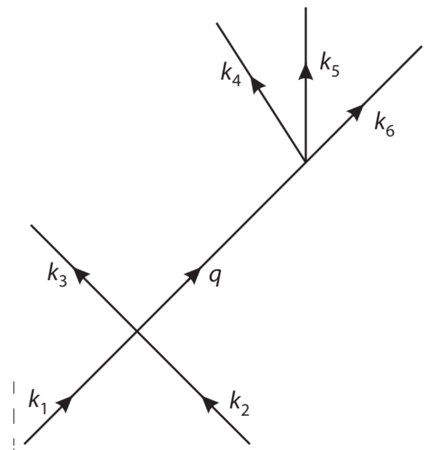

Figure 3

By rule 7, we don't need to carry around the δ function we get at the end, so we can just write the final amplitude as

$$\mathcal{M} = (-i\lambda)^2 \frac{i}{(k_4 + k_5 + k_6)^2 - m^2 + i\varepsilon}.$$

Much of the time we can just read off the final answer, without needing to compute the integral explicitly.

Already this tells us something new: the amplitude decreases as the momentum of the external line, $k_4 + k_5 + k_6$, gets further away from the mass m. The amplitude associated with the internal, "virtual" particle is penalized in proportion to how "unreal" it is.

Let's apply our model to a stranger situation, represented by the diagram in figure 4. Following our rules (and imposing momentum conservation immediately), we obtain the amplitude (ignoring proportionality constants):

$$\mathcal{M} \propto (-i\lambda) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k_1 + k_2 - k)^2 - m^2 + i\varepsilon}.$$

The integrand here is large if and only if one or both of the internal virtual particles is close to being real, as above. But notice how the integrand scales with k: it actually goes as $\frac{1}{k^4}$. $\int d^4k \ \frac{1}{k^4}$ diverges, and so does the above integral.

In fact, similar problems will show up whenever we have loops. But our theory can be salvaged! The details are outside of the scope of this paper,

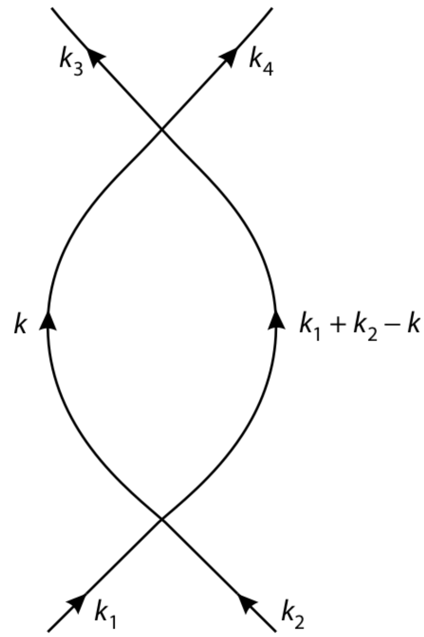

Figure 4

but there is a sense in which this infinity is not "real", and can be subtracted off, in a similar way to subtracting a constant. This procedure is called *renormalization.*

Finally, we can explore another interesting example. Consider one of the dropped terms from equation (5), which is of the form

$$-i\lambda \int_a \int_b \int_e \int_f D_{ae} D_{bf} J_a J_b J_e J_f \left(\int_w D_{ww} D_{ww} \right).$$

We again let J be the four delta functions, and we find a term proportional to

$$\left[-i\lambda D_{13} D_{24} \int_w D_{ww} D_{ww} \right] J_1 J_2 J_3 J_4.$$

plus terms obtained by permuting indices.

What is the physical interpretation of our result? Figure (5) holds the answer. We have a particle going from x_1 to x_3 and then disappearing, and similarly a particle going from x_2 to x_4 and then disappearing. Then we also

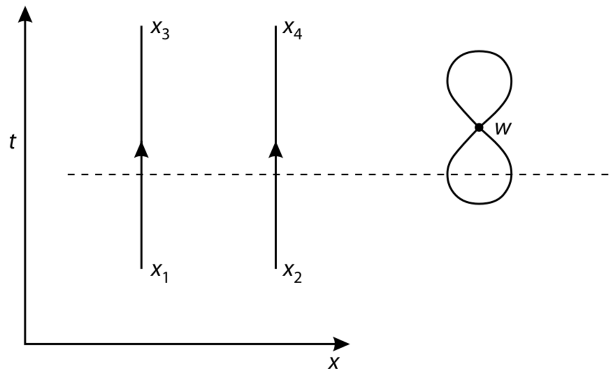

Figure 5

have a vertex that could be anywhere in spacetime, at which (with amplitude $-i\lambda$) two particles are created and the destroyed.

These are known as *vacuum fluctuations*, and it appears that they can happen spontaneously. There can be widely varying numbers of these particles at any one time. On the dotted line, for example, which represents a single moment, there appear to be four particles, but at other times, there are two, or even zero.

You may object: "Four particles? But there are only three lines!" This is again beyond the scope of this paper, but suppose we drew an arrows on the loops, and interpreted that just the same way we do in any other diagram: a particle traveling from one point to another (or in this case, one spacetime point back to itself). But time points upwards in this diagram, so what are we to make of the half of the line that seems to be traveling in the wrong direction? Well, it seems easy enough to say that it is the same particle, but traveling backwards in time! In fact, this leads us to another interpretation of antimatter.

As we have seen, Feynman diagrams are an invaluable tool for evaluating the amplitude of large classes of particle interactions. Not only do they provide essential quantitative information, they also tell us qualitative results that we would not have expected *a priori*. Thus, their central role in much of modern particle physics and quantum mechanics.

References

[1] Zee, A. *Quantum Field Theory in a Nutshell*, 1st ed., Princeton University Press, 2010, pp. 43-60.