Non-Equillibrium Universal Evolution

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In the early universe, particle interactions would occur at high rates, which kept most particle pairs in constant equilibrium. As the universe cooled there were several points where reaction rates dropped low enough for equilibrium to be broken. These points are of significant interest to cosmologists. We begin with an exploration of the Boltzmann equation, and follow by investigating three cases of interest: Big bang nucleosynthesis, protons and electrons forming neutral hydrogen, and dark matter formation.

1 The Boltzmann Equation

Suppose we have a pair of particles 1 and 2, which can annihilate, producing particles 3 and 4 (e.g. an electron and a proton forming a neutron and an electron neutrino). The reverse process can also occur, so we can write this reaction as $1+2 \leftrightarrow 3+4$. The Boltzmann equation in an expanding universe is:

$$\begin{aligned} a^{-3}\frac{d\left(n_1 a^3\right)}{dt} = &\int \frac{d^3p_1}{(2\pi)^3 2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\int \frac{d^3p_3}{(2\pi)^3 2E_3}\int \frac{d^3p_4}{(2\pi)^3 2E_4} \\ &* (2\pi)^4\delta^3(p_1+p_2-p_3-p_4)\delta(E_1+E_2-E_3-E_4)|\mathcal{M}|^2 \\ &* (f_3 f_4[1\pm f_1][1\pm f_2] - f_1 f_2[1\pm f_3][1\pm f_4]) \end{aligned} \tag{1}$$

If no reactions occur, the right-hand side of this equation is constant, meaning that number density falls off as a^{-3} as the universe expands. We now examine the right-hand side from the bottom line up. The last line, if we ignore the $1 \pm f$ terms for now, this line indicates that the rate of generation for species 1 is proportional to $f_3$and f_4, the occupation numbers for species 3 and 4. The $1 \pm f$ terms, where plus is used for Bosons and minus for Fermions, account for Bose enhancement and Pauli exclusion. It should be noted that the occupation numbers f_i are in fact dependent on the respective momenta p_i. On the second line the Dirac δ's enforce energy and momentum conservation, and the factors of 2π drop out of turning Kronecker δ's into Dirac δ's. $\mathcal{M}$ is a constant dependent on the specifics of the reaction. Typically it has the same value for $1+2 \rightarrow 3+4$ and $3+4 \rightarrow 1+2$. The top line integrates over all momenta. The factors of $(2\pi)^3$ are actually $(2\pi\hbar)^3$, and represent one unit of volume in phase space. The 2E terms come from the fact the integrals *should* be done over all four components of the momentum 4-vector, but are constrained on the sphere $E^2 = p^2 + m^2$.

The full Boltzmann equation, as given in Eq. 1, is certainly very complicated, but we can simplify it somewhat via certain approximations. Scattering processes will follow kinetic equilibrium, meaning the distributions will be either Bose-Einstein or Fermi-Dirac. This allows us to put all our potentials into a single variable μ. For reactions in equilibrium, μ is simply the chemical potential. For reactions *not* in chemical equilibrium, μ is the solution to an ordinary differential equation.

In the systems which we will be examining, we typically have $T < E - \mu$, so the exponential terms in the Bose-Einstein and Fermi-Dirac distributions dominate the ± 1. This means both distributions can be approximated as:

$$f(E) \approx e^{\mu/T} e^{-E/T} \tag{2}$$

We can also drop the $[1 \pm f]$ terms in the last line of Eq. 1, so instead we can write:

$$\begin{aligned} &(f_3 f_4 [1 \pm f_1][1 \pm f_2] - f_1 f_2 [1 \pm f_3][1 \pm f_4]) \\ &\approx e^{-(E_1+E_2)/T} \left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right) \end{aligned} \tag{3}$$

Where energy conservation, i.e. $E_1 + E_2 = E_3 + E_4$ is assumed. In place of μ we use number density, which is related to μ by:

$$n_i = g_i e^{\mu_i/T} \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} \tag{4}$$

Where g_i is the species degeneracy. We also define equilibrium number density as:

$$n_i^{(0)} \equiv \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T} \tag{5}$$

Which allows us to rewrite the last line of Eq. 1 again as:

$$\begin{aligned}\langle \sigma\nu \rangle \equiv & \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} e^{-(E_1+E_2)/T} \\ & * (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4)\delta(E_1 + E_2 - E_3 - E_4)|\mathcal{M}|^2\end{aligned} \tag{6}$$

So the Boltzmann equation further reduces to:

$$a^{-3} \frac{d\left(n_1 a^3\right)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma\nu \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right) \tag{7}$$

Lastly for this section we introduce a useful concept. When reaction rates are high, we have

$$\frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} = \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} \tag{8}$$

This relation is called by a number of names, such as *chemical equilibrium*, *nuclear statistical equilibrium*, or *the Saha equation*.

2 Big Bang Nucleosynthesis

When the universe's temperature was roughly 1MeV, the cosmic plasma was composed of:

- Relativistic particles in equilibrium (photons, electrons, and positrons): These are kept in equilibrium by EM interactions, and have mostly the same abundances.

- Decoupled relativistic particles (neutrinos): Neutrinos are uncoupled from the cosmic plasma at this time, but keep roughly the same energy.

- Nonrelativistic particles (baryons): Due to the initial baryon/anti-baryon asymmetry, all anti-baryons have by this point annihilated, leaving a baryon abundance of:

$$\eta_b = \frac{n_b}{n_\gamma} = 5.5 \times 10^{-10} \left(\frac{\Omega_b h^2}{.02} \right) \tag{9}$$

As energy drops nuclear reactions do not occur fast enough to keep the particles in equilibrium. To avoid having to solve the Boltzmann equation for all possible nuclei, we can make a few simplifications. Firstly that Hydrogen and Helium are the only elements produced in considerable amounts, and secondly that above .1 MeV no nuclei form at all, allowing us to solve for neutron/proton ratio and use that as a starting point. Both these simplifications follow from the fact the photon:baryon ratio is very high, so all nuclei that form are split apart by a photon not long after.

2.1 Neutron Abundance

Protons can be converted to neutrons (and vise-verse) via weak interactions such as $p + e^- \leftrightarrow n + \nu_e$. These reactions keep neutrons and protons in equilibrium until T 1 MeV. Afterward we have:

$$\frac{n_p^{(0)}}{n_n^{(0)}} = \frac{e^{m_p/T} \int dpp^2 e^{-p^2/2m_p T}}{e^{m_n/T} \int dpp^2 e^{-p^2/2m_n T}} \tag{10}$$

The integrals are proportional to $(m_p/m_n)^{3/2}$, which is approximately 1, so we can discard this term. We define $\mathcal{Q} = m_p - m_n$, so

$$\frac{n_p^{(0)}}{n_n^{(0)}} = e^{\mathcal{Q}/T} \tag{11}$$

This means that the ratio gets larger as temperature drops. We define

$$X_n \equiv \frac{n_n}{n_n + n_p} \tag{12}$$

So X_n is the ratio of neutrons to total baryons. We define equilibrium ratio:

$$X_{n,EQ} \equiv \frac{1}{1 + \left(n_p^{(0)}/n_n^{(0)} \right)} \tag{13}$$

So we can fill out the Boltzmann equation where 1 is neutron, 3 is proton, and 2 and 4 are leptons in equilibrium:

$$a^{-3}\frac{d\left(n_n a^3\right)}{dt} = n_l^{(0)}\langle\sigma\nu\rangle\left[\frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n\right] \tag{14}$$

We make the identification $n_l^{(0)}\langle\sigma\nu\rangle = \lambda_{np}$, and make the substitution $n_n = (n_n + n_p)X_n$ to rewrite this as:

$$\frac{dX_n}{dt} = \lambda_{np}\left[(1 - X_n)e^{\mathcal{Q}/T} - X_n\right] \tag{15}$$

We further define $x \equiv \mathcal{Q}/T$, so the left-hand side becomes $(dX_n/dx)(dx/dt)$. Because $T \propto a^{-1}$, we have

$$\frac{1}{T}\frac{dT}{dt} = -H = -\sqrt{\frac{8\pi G\rho}{3}} \tag{16}$$

As nucleosynthesis occurs in the radiation era, ρ comes largely from relativistic particles, so

$$\rho = \frac{\pi^2}{30}T^4 g_* \tag{17}$$

Where g_* is effective relativistic degrees of freedom, which is dependent on temperature. For the period of interest $g_* \approx 10.75$. We can then rewrite the Boltzmann equation again, this time as

$$\frac{dX_n}{dx} = \frac{x\lambda_{np}}{H(x=1)}\left[e^{-x} - X_n\left(1 + e^{-x}\right)\right] \tag{18}$$

where

$$H(x=1) = \sqrt{10.75\frac{4\pi^3 G\mathcal{Q}^4}{45}} = 1.13s^{-1} \tag{19}$$

We also need an approximation for the conversion rate λ_{np}. Under our approximations we have:

$$\lambda_{np} = \frac{255}{\tau_n x^5}\left(12 + 6x + x^2\right) \tag{20}$$

Where $\tau_n = 886.7s$ is neutron lifetime. When $T = \mathcal{Q}$, $x = 1$, the conversion rate will be 5.5 s^{-1}, above the expansion rate. As T drops x increases, so λ_{np}

drops as x^5, rapidly falling below the expansion rate. Below .1 MeV neutron decay ($n \rightarrow p + e^- + \bar{\nu}$) and deuterium production ($n + p \rightarrow D + \gamma$) become significant. Taking these into account we have that the neutron fraction when nucleosynthesis begins is:

$$X_n(T_{nuc}) = 0.11 \tag{21}$$

2.1.1 Light Elements

We can approximate light element production as occuring instantly at $T = T_{nuc}$. Consider deuterium for example. If the universe were always in equilibrium, all baryons would eventually form deuterium. So we have:

$$\ln(\eta_b) + \frac{3}{2}\ln\left(\frac{T_{nuc}}{m_p}\right) \sim -\frac{B_D}{T_{nuc}} \tag{22}$$

Which suggests that deuterium production occurs at $T \approx .07 MeV$, and depends weakly on $\ln(\eta_b)$. Since helium has greater binding energy than deuterium, it is favored and almost all remaining neutrons become ^{4}He. As each helium has two neutrons, this gives us:

$$X_4 \equiv \frac{4n_{^4He}}{n_b} = 2X_n(T_{nuc}) \tag{23}$$

3 Recombination

At $\sim$ 1 eV, photons, electrons and, protons remain tightly coupled, so at this point in time there is very little neutral hydrogen. Any that does form is quickly ionized by the high number of photons. While the reaction $e^- + p \leftrightarrow \mathrm{H} + \gamma$ is in equilibrium, we have the relation:

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \tag{24}$$

Since the universe is neutrally charged, we also have $n_e = n_p$, so we can define electron fraction as:

$$X_e \equiv \frac{n_e}{n_e n_H} = \frac{n_p}{n_p n_H} \tag{25}$$

Evaluating the integrals hidden in Eq. 24, we get

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_e + n_H}\left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-[m_e+m_p-m_H]/T}\right] \tag{26}$$

Where we neglect the mass difference between H and m_p in the non-exponential terms. The term in the exponential is ϵ_0/T (Where ϵ_0 is hydrogen binding energy, not free space permeability). Since helium is only produced in small amounts, we can approximate the total nuclei density $n_p + n_H$ by total baryon density, $\eta_b n_\gamma \sim 10^{-9}T^3$. When T is on the order of ϵ_0, the right-hand side is of order 10^15, which requires X_e to be approximately 1, meaning almost all hydrogen is ionized. As T falls the fraction of neutral hydrogen rises, but does so out of equilibrium, so we must solve the Boltzmann equation for the electron density:

$$\begin{aligned} a^{-3}\frac{d(n_e a^3)}{dt} &= n_e^{(0)} n_p^{(0)} \langle\sigma\nu\rangle \left[\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}}\right] \\ &= n_b\langle\sigma\nu\rangle\left[(1-X_e)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} - X_e^2 n_b\right] \end{aligned} \tag{27}$$

We can define the ionization rate β and recombination rate $\alpha^{(2)}$ by:

$$\beta \equiv \langle\sigma\nu\rangle\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T} \tag{28}$$

$$\alpha^{(2)} \equiv \langle\sigma\nu\rangle \tag{29}$$

Where the superscripted (2) reflects the fact that any ionization to the ground state will produce a photon that will ionize another atom, leading to no net change in X_e. With these identifications we can rewrite Eq. 27 as:

$$\frac{dX_e}{dt} = \left[(1-X_e)\beta - X_e^2 n_b \alpha^{(2)}\right] \tag{30}$$

We can obtain the time evolution of X_e in detail via numerical integration of this equation. Free electron abundance is highly relevant to observational cosmology, because it affects the anisotropies is backgroud radiation observable today through rates of decoupling. Decoupling occurs when Compton scattering begins to happen at a lower rate than the expansion rate. The scattering rate is given by

$$n_e\sigma_T = X_e n_b \sigma_T \tag{31}$$

Where σ_T is the Thompson cross-section. The ratio of the baryon density n_b to the critical density ρ_{cr} is given by $m_p n_b/\rho_{cr} = \Omega_b a^{-3}$, so we can put n_b in terms of Ω_b:

$$n_e\sigma_T = 7.477 \times 10^{-30}\text{cm}^{-1} X_e \Omega_b h^2 a^{-3} \tag{32}$$

So if we divide by the expansion rate we get:

$$\frac{n_e\sigma_T}{H} = .0692 a^{-3} X_e \Omega_b h \frac{H_0}{H} \tag{33}$$

At early times the main contribution to the Hubble rate is either matter or radiation, so $H/H_0 = (\Omega_m)^{1/2} a^{-3/2}[1 + a_{eq}/a]^{1/2}$, so our final equation is:

$$\frac{n_e\sigma_T}{H} = 113 X_e \left(\frac{\Omega_b h^2}{.02}\right)\left(\frac{.15}{\Omega_m h^2}\right)^{1/2}\left(\frac{1+z}{1000}\right)^{3/2}\left[1 + \frac{1+z}{3600}\frac{.15}{\Omega_m h^2}\right]^{-1/2} \tag{34}$$

Photons decouple when X_e drops below 10^{-2}, so this occurs during recombination.

4 Dark Matter

Observational evidence suggests the existence of non-baryonic dark matter with $\Omega_{dm} \approx .3$. The most popular candidate is some sort of weakly-interacting massive particle, or WIMP. According to the theory, WIMPs interacted with the cosmic plasma in the early universe, but failed to keep in equilibrium as temperature dropped. Our goal will be to solve the Boltzmann equation for such a particle, to find what mass and cross-section will produce $\Omega_{dm} = .3$.

The generic WIMP model goes as follows: Consider the interaction between two heavy particles X, which annihilate and produce two nearly massless particles l. The l's remain coupled to the plasma, which keeps them in equilibrium, $n_l = n_l^{(0)}$. We can then use the Boltzmann equation to solve for the abundance of particle X, n_X:

$$a^-3\frac{d(n_X a^3)}{dt} = \langle\sigma\nu\rangle\left[\left(n_X^{(0)}\right)^2 - n_X^2\right] \tag{35}$$

As T typically scales with a^{-1}, we can rewrite the left-hand side as $T^3 d(n_X/T^3)/dt$. We define a new variable

$$Y \equiv \frac{n_X}{T^3} \tag{36}$$

Giving us a differential equation in terms of Y:

$$\frac{dY}{dt} = T^3 \langle \sigma \nu \rangle [Y_{EQ}^2 - Y^2] \tag{37}$$

Where $Y_{EQ} \equiv n_X^{(0)}/T^3$ We can reparameterize in terms of a variable $x \equiv m/T$, where m is the mass of X. We use the Jacobian $dx/dt = Hx$ to make this change. Dark matter is produced mostly in the radiation era, when energy density scales with T^4, so $H = H(m)/x^2$, so our equation becomes:

$$\frac{dY}{dx} = -\frac{\lambda}{x^2} \left(Y^2 - Y_{EQ}^2 \right) \tag{38}$$

Where λ is defined as:

$$\lambda \equiv \frac{m^3 \langle \sigma \nu \rangle}{H(m)} \tag{39}$$

Many theories hold λ to be constant, though in some it has temperature dependence. The following calculations assume the former case.

The differential equation we are interested in has no analytic solution in general. However we can analytically find the final abundance after the freeze-out. We define $Y_\infty \equiv Y(x = \infty)$. For $x \sim 1$, the left-hand side of Eq. 37 is of order Y, while the right-hand side is of order λY^2, which as λ is typically large forces $Y_{EQ} \approx Y$. After the freeze-out Y_{EQ} drops quickly, as X falls out of equilibrium. So for late times we can approximate Eq. 37 as

$$\frac{dY}{dx} \approx -\frac{\lambda Y^2}{x^2}$$

Which we can integrate analytically from the freeze-out time x_f to $x = \infty$, which gives us:

$$\frac{1}{Y_\infty} - \frac{1}{Y_f} = \frac{\lambda}{x_f} \tag{40}$$

As Y_f is typically far greater than Y_∞, we can approximate this as

$$\frac{1}{Y_\infty} \approx \frac{\lambda}{x_f} \tag{41}$$

The freeze-out time x_f is typically thought to be of order 10, a good enough approximation for our purposes.

After the freeze-out, density of X falls off as a^{-3}, so its energy density is $m(a_1/a_0)^3$, where a_1 is a time sufficiently late such that $Y = Y_\infty$. The number density is then $Y_\infty T_1^3$ (by definition of Y), so we have

$$\rho_X = m Y_\infty T_0^3 \left(\frac{a_1 T_1}{a_0 T_0}\right)^3 \approx \frac{m Y_\infty T_0^3}{30} \tag{42}$$

Finally to obtain Ω_X we simply divide our expression for Y_∞ by ρ_{cr}:

$$\begin{aligned} \Omega_X &= \frac{x_f}{\lambda} \frac{m T_0^3}{30 \rho_{cr}} \\ &= \frac{H(m) x_f T_0^3}{30 m^2 \langle \sigma \nu \rangle \rho_{cr}} \\ &= \left[\frac{4\pi^3 G g_*(m)}{45}\right]^{1/2} \frac{x_f T_0^3}{30 \langle \sigma \nu \rangle \rho_{cr}} \end{aligned} \tag{43}$$

Note that Ω_X has no explicit m-dependence, so we only need to solve for cross-section. In the eras of interest, g_* takes contributions from all standard model particles, and is thus of order 100. Normalizing g_* and x_f we get

$$\Omega_X = .3 h^{-2} \left(\frac{x_f}{10}\right) \left(\frac{g_*(m)}{100}\right)^{1/2} \frac{10^{-39}\text{cm}^2}{\langle \sigma \nu \rangle} \tag{44}$$

Which suggests a cross section of order 10^{-39}, which is predicted by several theories.

References

This paper, for the most part, directly follows Chapter 3 of Scott Dodelson's *Modern Cosmology.*