Collisional Boltzmann Equation

Need to modify purely collisionless Boltzmann equation to include interactions between particles

We consider the case of a dilute gas

$n a^3 << 1$ (small particle radius $= a$ compared to interparticle spacing).

and no long-range interactions between particles.

Now the collisionless Boltzmann equation says that $f(\vec{x}, \vec{p}, t)$ does not change along the trajectory of a particle. Collisions can change this by bumping particles to different velocities, thus increasing or decreasing the number of particles in a given μ-space

thus.

$$\frac{Df}{Dt} d^3x\, d^3u = C_{in} - C_{out}$$

$C_{in}, C_{out} \equiv$ rates at which particles enter or leave $d^3x\, d^3u$ from collisions we consider elastic collisions:

$2u_1'u_0' = 2u_1u$

$$\vec{u} + \vec{u}_1 = \vec{u}' + \vec{u}_1' \qquad \text{momentum cons}$$

( )

(mom)

$\vec{u}, \vec{u}_1$ = particle velocities before collisions

$\vec{u}', \vec{u}_1'$ = velocities after collision

$(u_1 - u_2)^2 = (u_1' - u_2')^2$

$u_1^2 + u_2^2 = u_1'^2 + u_2'^2$

$-2u_1u_2 + 2u_1'u_2'$

$$\frac{1}{2}|\vec{u}|^2 + \frac{1}{2}|\vec{u}_1|^2 = \frac{1}{2}|\vec{u}'|^2 + \frac{1}{2}|\vec{u}_1'|^2 \qquad \text{energy cons}$$

$(u_1 - u_0)^2 = (u' - u_1')^2$

these equations provide 4 equations for 6 unknowns. ($\overset{3}{\vec{u}'}, \overset{3}{\vec{u}_1'}$) final velocities

The remaining constraints come from:

1) coplanarity of $\vec{u}', \vec{u}_1', \vec{u}, \vec{u}_1$ for radial force of interactions (e.g. coulomb collisions) - eliminates

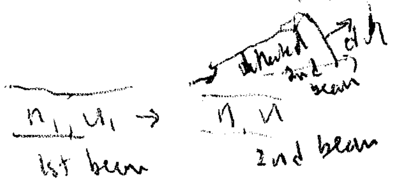
2) impact parameter, which gives the ∡ of deflection. This comes from microphysics of interaction

Statistically, # 2) is modeled by differential cross section. We assume its given and show how dynamics of system can then be studied:

consider beam of particles with number density n_1 and velocity $\vec{u}_1$, colliding with beam having number density n and velocity $\vec{u}$. The latter beam sees particle flux $I = |\vec{u} - \vec{u}_1| n_1$ from first beam

(number per area per time)

Define $\delta n_c \equiv \dfrac{\#\ \text{collisions}}{\text{time} \cdot \text{volume}}$ that deflect particles from second beam into solid angle $d\Omega$, by interaction with first beam: (18)



$$\delta n_c = \underbrace{(n)}_{\substack{n \text{ of second} \\ \text{beam}}} \underbrace{(I)}_{\substack{\text{flux of } n_1 \text{ that} \\ n\text{-beam is exposed to}}} \underbrace{d\Omega}_{\text{solid } \measuredangle} \underbrace{\sigma(\vec{u}, u_1 | u', u_1')}_{\substack{\text{differential scattering} \\ \text{cross section}}}$$

$\delta n_c = \dfrac{\#}{\text{vol, time, sol.} \measuredangle}$

(individual interactions are reversible for elastic scattering so that

$$\sigma(\vec{u}, u_1 | u_1', u') = \sigma(\vec{u}_1', \vec{u}' | \vec{u}, \vec{u}_1) \;)$$

Now

since $n = f(\vec{x}, \vec{u}, t)\, d^3\vec{u}$ = number per volume

and $n_1 = f(\vec{x}, u_1, t)\, d^3 u_1$

and $I = |\vec{u} - \vec{u}_1| n_1 = |u_1 - u_1| f(\vec{x}, \vec{u}_1, t)\, d^3 u_1$

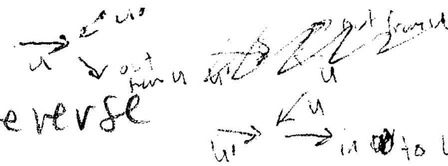
$$\delta n_c = \sigma(u, u_1 | u', u_1') |\vec{u} - \vec{u}_1| f(x, u, t) f(x, u_1, t)\, d\Omega\, d^3u\, d^3u_1$$

since $C_{out} = \dfrac{\#\ \text{collisions}}{\text{time}}$ in phase volume $d^3x\, d^3u$,

$$C_{out} = d^3x \underbrace{\int \delta n_c (d\Omega\, d^3u_1)}_{\text{means } d^3x \int_{d\Omega,\, d^3u_1} \delta n_c} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(u, u_1 | u', u_1') |\vec{u} - \vec{u}_1| f(x, \vec{u}, t) f(x, \vec{u}_1, t)$$

i.e. multiply δn_c by d^3x and integrate over d^3u_1, $d\Omega$

→



To get C_{in} consider reverse collisions; that is replace $u' \leftrightarrow u$ and $u_1' \leftrightarrow u_1$ straight away we have:

$$C_{in} = d^3x\, d^3u' \int d^3u_1' \int d\Omega\, \sigma(u,u_1|u',u_1')|u'-u_1'| f(x,u',t) f(x,u_1',t)$$

But: ① conservation of momentum & energy for collisions $\Rightarrow$ $|u-u_1| = |u'-u_1'|$

and ② Earlier we proved (eqn 13b) that phase space measures at any time are equal (from Liouville's Thm + conservation of particle number & those elastic collision assumption) thus for 2-particle space

$d^3u\, d^3u_1 = d^3u'\, d^3u_1'$. ③ we also argued $\sigma(u,u_1|u',u_1') = \sigma(u',u_1'|u,u_1)$. Thus ①, ②, ③

$$\Rightarrow C_{in} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(u',u_1'|u,u_1)|u-u_1| f(x,u',t) f(x,u_1',t)$$

comparing to C_{out} we then combine to get:

$$\frac{Df}{Dt}\, d^3x\, d^3u = C_{in} - C_{out} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)(f' f_1' - f f_1)$$

(where $f' \equiv f(u')$ and $f_1' \equiv f(u_1')$
$f_1 \equiv f(u)$; $f' = f(u')$) $\longrightarrow$

we thus have

$$\boxed{\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \vec{u}\cdot\nabla f + \frac{\vec{F}}{m}\cdot\nabla_u f = \int d^3u_1 \int d\Omega\, |u-u_1|\,\sigma(\Omega)\,(f'f_1' - ff_1)}$$

$\vec{F} = m\dot{\vec{u}}$ is any force field that particles experience e.g. gravity

Collisional Boltzmann eqn (14)

to recap: right side measures effects of collisions on distribution function for a dilute gas. (dilute because we assumed only binary collisions)

Maxwellian Distribution

Uniform classical gas relaxes to maxwell dist. This can be derived from above collisional Boltz. eqn: Consider case when $\vec{F}$ term is negligible, and f is independent of time and space (ie. in equilibrium).

Boltz eqn ⇒

$$f f_1 = f' f_1'$$

or

$$\log f(u) + \log f(u_1) = \log f'(u) + \log f_1'(u_1)$$

→

If $\chi(u)$ is a conserved quantity

then $\underbrace{\chi(u) + \chi(u_1)}_{\text{before}} = \underbrace{\chi(u') + \chi(u_1')}_{\text{after collision}}$

Since this has same form of previous equation we must be able to write $\log f(u)$ as a linear combination of $\chi(u)$

That is:

$$\log f(u) = C_0 + \underbrace{\sum_s}_{\text{sum over all conserved quantities}} C_r \chi_s(\vec{u}) \qquad (C_0, C_r \text{ are constants})$$

as energy & the 3 momenta are the (complete) relevant quantities here:

$$\log f(\vec{u}) = C_0 + C_1 \vec{u}^2 + C_{2x} u_x + C_{2y} u_y + C_{2z} u_z$$

$$\Rightarrow \log f(\vec{u}) = -B(\vec{u} - \vec{u}_0)^2 + \log A$$

where $C_0, C_1, C_{2x}, C_{2y}, C_{2z}$ have been replaced by

exponentiate $\quad B, A, u_{0x}, u_{0y}, u_{0z},$

$$\Rightarrow f(u) = A e^{-B(\vec{u} - u_0)^2}$$

$$n = \int d^3u f(u) \Rightarrow A = \left(\frac{B}{\pi}\right)^{3/2} n$$

$\rightarrow$

$\Rightarrow$

$$f(u) = \left(\frac{B}{\pi}\right)^{3/2} n\, e^{-B(\vec{u}-u_0)^2}$$

note that

$$\langle \vec{u} \rangle = \frac{1}{n}\int f(u)\,\vec{u}\, d^3u = \left(\frac{B}{\pi}\right)^{3/2}\int d^3\vec{u}\,(u+U_0)\, e^{-Bu^2}$$

(where $\vec{u} \to \vec{u} + \vec{U}_0$ change of variables was used)

$$= \vec{U}_0 \left(\frac{B}{\pi}\right)^{3/2}\int d^3u\, e^{-Bu^2} = \vec{U}_0$$

$\Rightarrow$ non-zero $\vec{U}_0$ implies a <u>mean</u> streaming motion.

$$I_n = \int_{-\infty}^{\infty} e^{-Bu^2} u^n\, du$$

$$I_{n+2} = -\frac{\partial I_n}{\partial B}$$

$$I_0 = \sqrt{\frac{\pi}{B}}$$

$$I_2 = \frac{1}{2}\sqrt{\frac{\pi}{B^3}}$$

if we go to frame in which $\vec{U}_0 = 0$ and consider system of temperature T, then $B = \frac{m}{2k_BT}$

and

$$f(u) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{m\vec{u}^2}{2k_BT}\right]$$

<u>Maxwell Boltzmann</u> is a soln to steady-state Boltzmann equation

NOT SURPRISING!