...s number, and dimensionless scaling relations

...odel of a plane or car or astrophysical jet
scaled ...own to "table top" size appropriately
model the dynamics of the real thing?

power of dimensionless numbers

consider object of size L velocity U
thus characteristic time is $\sim L/U$
let x', v', t', w' be dimensionless units normalized
to these values. Then:

$$x = x'L, \quad v = v'U, \quad t = t'\frac{L}{U}, \quad w = w'\frac{U}{L} \tag{80}$$

recall that for incompressible flows

$$\frac{\partial w}{\partial t} = \nabla \times (v \times w) + \nu \nabla^2 w,$$

then, using (80) we can write

$$\frac{\partial w'}{\partial t} = \nabla \times (v' \times w') + \frac{1}{R} \nabla^2 w' \tag{81}$$

where $R = \frac{LU}{\nu}$ is the Reynolds number note that ν has units of $\frac{\text{length}^2}{\text{time}}$ so R is dimensionless.

This is important: for two systems with the same R, the behavior as governed by (81). Thus to properly model astrophysical flows, or planes etc. in the lab, one must do experiments with same R.

For $R \gtrsim 3000$, (using L as radius of pipe and u as velocity of mean flow) flow through pipe is unstable to becoming turbulent. $R < 3000$, flow through pipe is laminar.

Note also that R appears to indicate the relative importance of the last two terms in (81) but this is not always quite right! → →

For $R \ll 1$, (81) becomes

$$\nabla^2 w' = 0$$

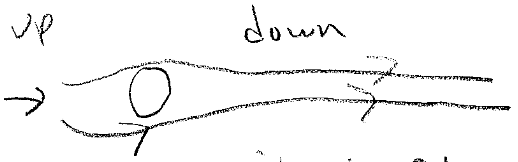
Stokes (1851) showed that a sphere of radius a moving through a viscous fluid with velocity u, density ρ, viscosity ν incurs drag force of $F_D = 6\pi \rho \nu a u$ this is called Stokes Law for viscous flows.

Notice that for $R \gg 1$, it would appear from (81) that the viscous term (the last term) can be ignored, and one might expect the system to be approximated by an ideal fluid. But it is more complicated in reality, when experiments are performed to test the drag force:

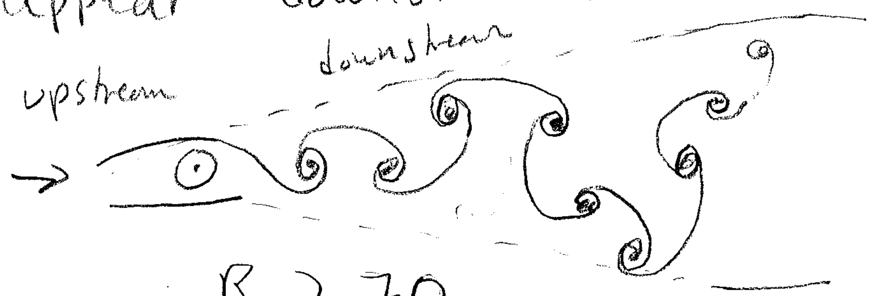
$\rightarrow$

Flow past a cylinder for $30 > R > 10$

- looks like ideal flow but for $R > 30$ vortices begin to appear downstream.



ideal flow



$R > 70$

The vortices appear in a "wake" that increases in width farther downstream. (= "karman vortex sheet")

At very large R, the wake becomes turbulent, flow has large random velocities → not like ideal flow at all! what is going on to produce this highly non-ideal behavior, despite large R?

First, note than when turbulent wake is present, drag on cylinder or sphere much larger than Stokes Law:

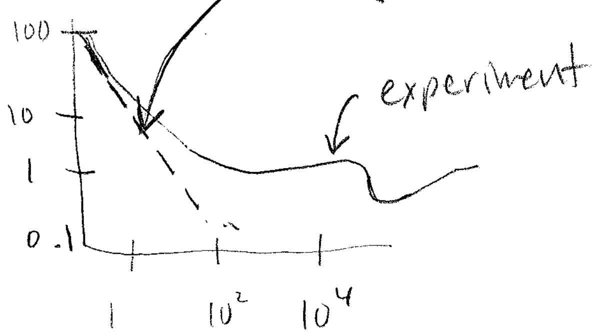
In the large R regime:

$$F_D \simeq C_D (\pi a^2) \frac{\rho U^2}{2} \qquad (82)$$

where C_D can be measured →

If the drag force always equaled the stokes value then setting (82) equal to stokes drag =>

$C_D = \frac{12}{R}$, but experimentally:



the drag coefficient falls off much more slowly at large R

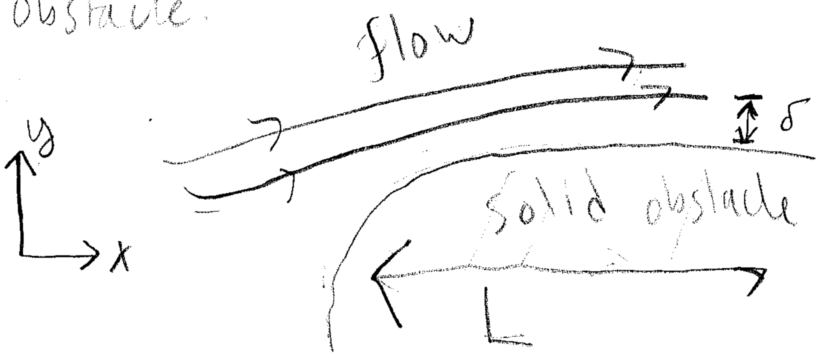
The reason has to do with <u>boundary layers</u>

Near to the surface of the obstacle in the flow, velocity must change from large values to zero. Since this happens over small scales, the effective R in that region is not much greater than 1, so near to the obstacle's surface the flow is far from ideal.

the $\nu \nabla^2 v$ term in the Navier-Stokes equation becomes important because v changes on scale $\delta \ll a$, so

$$R_{eff} \equiv \frac{V\delta}{\nu} \ll R \equiv \frac{Va}{\nu} \quad \rightarrow$$

We can see also that the boundary layer grows with distance behind the obstacle:



Navier-Stokes equation for $\vec{V}_x$ is given by:

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho} \partial_x P + \nu \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} \right)$$

Assuming $V_y \lesssim V_x \, \delta/L$ and since P, V_x are not expected to vary much along x, the dominant terms are

$$V_x \frac{\partial V_x}{\partial x} \backsim \nu \frac{\partial^2 V_x}{\partial y^2}$$

or to order of magnitude

$$\frac{V_x^{\cancel{2}}}{L} \backsim \frac{\nu \cancel{V_x}}{\delta^2} \Rightarrow \boxed{\delta = \sqrt{\frac{\nu L}{V_x}}}$$

boundary layer grows as square root of distance downstream!

$\rightarrow$

Note that because the viscosity is important in the boundary layer, Kelvin's vorticity theorem is violated. Thus flux of vorticity is <u>not conserved</u> there and new vortex lines can form → that explains why vortices can develop & grow in the turbulent wake.

The reason for the development of a turbulent boundary layer is shear instabilities that develop at the sides of the obstacle from strong velocity gradients (conditions for these instabilities can be derived) the turbulence is then carried downstream. Since the turbulence is a randomization of the bulk velocity, which eventually dissipates as heat, some of the bulk energy of motion of the obstacle is lost ⇒ this is why turbulence produces a drag! Equivalently, one can think of the bulk flow energy being randomized, if object is at rest.

"Order of magnitude" estimates for Stokes, Reynolds & Epstein Drag

Stokes Drag: when the flow is laminar and the object size is much larger than the mean free path of particles, the drag force F_d must depend on the flow velocity u, the object size, a, the flow density ρ, and the viscosity ν. But the only combination of these quantities that produces units of force is $\sim \rho u \nu a \sim \underbrace{\left[\frac{\text{mass}}{\ell^3}\,\frac{\ell}{t}\,\frac{\ell^2}{t}\,\ell\right]}_{\text{(units)}} = \left[\frac{m\,\ell}{t^2}\right] = \underbrace{[F]}_{\text{force units}}$

More detailed calculations produce

$F_d = 6\pi \rho u \nu a$ given earlier

Reynolds Drag - when the flow is fast enough that turbulence ensues, the drag no longer depends explicitly on the viscosity. Then one must construct a force with ρ, u, and a only: the combination that works is given by

$F_d \propto \rho u^2 \underbrace{\pi a^2}_{\text{area}} \sim \left[\frac{m}{\ell^3}\,\frac{\ell^2}{t^2}\,\ell^2\right] \sim \left[\frac{m\,\ell}{t^2}\right] \sim \underbrace{[F]}_{\text{force}}$

typically a drag constant is empirically measured:

$\Rightarrow F_d \sim \pi C_d \rho u^2 a^2$

Epstein Drag

when the mean free path λ_{mfp} is larger than the object size, then the drag is due to collisions with individual particles. In this case the particles collide with the object at speeds sampled from the particle distribution function $f(p,x,t)$. On average however, for a quasi-maxwellian distribution, the average particle speed is the sound speed. The drag force must depend on the object speed, the mass of the particles colliding with the object and the frequency at which this occurs. The frequency of collisions is

$$\sim \underbrace{n}_{\text{number density}} \underbrace{(\pi a^2)}_{\text{area of object}} \underbrace{(C_s - u)}_{\text{relative velocity}} \simeq n\pi a^2 C_s \text{ for subsonic } u \ll C_s \text{ flows.}$$

Combining this frequency with the particle mass and object speed to form a force requires multiplying by the particle mass and flow speed to obtain $F_{d,ep} \propto \underbrace{M_H}_{\text{hydrogen}} n\pi a^2 C_s u \sim \underbrace{\rho}_{\text{density of diffuse medium}} \pi a^2 C_s u$

More detailed calcs give $F_{d,ep} \sim 2\rho\pi a^2 C_s u$

another way to think of the drag force is that the time scale for the object to change its speed by an order of magnitude is roughly

$$\tau \sim \frac{u}{\frac{du}{dt}} \sim \omega_c^{-1} \frac{m_{obj}}{m_H} \qquad (*)$$

$\uparrow$ collision frequency in the case of Epstein drag.

The ratio of masses appears on the right side because, it takes of order 1 collision of an H atom to change the speed of an equivalent mass in the object. Thus we require $N = \frac{m_{obj}}{m_H}$ collisions to change the object speed.

But $(*)$ is the same as the force equation

$$m_{obj} \frac{du}{dt} = m_H u \omega_c \simeq \rho \pi a^2 c_s u$$

derived on the previous page.