ELECTRIC POTENTIAL
Summary of Last Lecture

Two methods have been presented for calculating the electric field.

**Coulomb’s Law:**
Discrete charges:

\[
\mathbf{E}_{\text{net}} = \sum_{i=1}^{n} \frac{1}{4 \pi \varepsilon_0} \frac{q_n}{r_{i0}^2} \mathbf{r}_{i0}
\]

Continuous charges:
In cartesian coordinates the \( \mathbf{E} \) field at the point \( p(xyz) \) is written as an integral over the charge distribution at \( p'(x'y'z') \):

\[
\mathbf{E}_{p(xyz)} = \frac{1}{4 \pi \varepsilon_0} \int_V \frac{\rho(x'y'z')}{r_{p'p}^2} dx'dy'dz' \mathbf{r}_{p'p}
\]

**Gauss’s Law**

\[
\Phi_{\text{net}} = \oint_{\text{surface}} \mathbf{E}_{\text{net}} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\text{enclosed}} \rho d\tau
\]

Gauss’s law combined with symmetry was shown to be a simple method for calculating electric fields for those systems having a symmetry. In particular, spherical, cylindrical and planar symmetry were considered.

Once the electric field has been computed, then it is possible to predict the motion of charged particles in this field using Newton’s laws of motion. The motion of an electron in an electric field was discussed.
Assumptions used to derive Gauss’s Law

Note that the assumptions used to derive this are:

- Coulomb’s Law
- Principle of Superposition

Note that the two crucial aspects of Coulomb’s law that lead to Gauss’s law are that the electric field for a point charge is:

- a) exactly proportional to $\frac{1}{r^2}$
- b) the field is radial.
Gauss’s Law ↔ Coulomb’s Law

- Gauss’s Law more fundamental, applicable to both static and moving charges
- Gauss’s Law is the first of Maxwell’s four equations
- Coulomb’s law only applicable for static point charges
Analogy between gravitation and electrostatics

Coulomb’s Law

\[ \vec{F}_{10} = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_0}{r_{10}^2} \vec{r}_{10} \]

\[ \vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{r} \]

Newton’s Law of Gravitation

\[ \vec{F}_{Grav}^{10} = -G \frac{m_1 m_0}{r_{10}^2} \vec{r}_{10} \]

\[ \vec{E}_{grav} = -G \frac{m}{r^2} \hat{r} \]

\[
\oint_{\text{surface}} \vec{E}_{\text{net}} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int_{\text{enclosed}} \rho d\tau
\]

\[
\oint_{\text{surface}} \vec{E}_{\text{grav}} \cdot d\vec{S} = -4\pi G \int_{\text{enclosed}} \rho_m d\tau
\]
Symmetry

Symmetry is a powerful concept for solving problems

Gauss’s Law:

\[ \Phi_{\text{net}} = \oint_{\text{surface}} \mathbf{E}_{\text{net}} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\text{enclosed}} \rho d\tau \]

- Symmetry often can be used to identify a Gaussian surface for which \( \mathbf{E} \) is uniform and perpendicular to the surface, this allows \( \mathbf{E} \) to factor out of the flux integral in Gauss’s Law providing a simple determination of \( \mathbf{E} \)

- Three major symmetries: spherical, cylindrical, planar
Electric Potential

• Electric potential energy, $U$

• Electric potential, $V$

• $V$ for a system of point charges

• $V$ for a continuous charge distribution

• Why determine electric potential?

• Determination of $E$ from electric potential $V$

• Circulation of a static electric field
Electric Potential Energy

In gravitation we learned that the concept of an energy associated with position, or potential energy, is a very useful concept. The potential energy difference $\Delta U_{a \rightarrow b}$ between two points $\vec{r}_a$ and $\vec{r}_b$, is the work done moving from a to b against a force $\vec{F}$. That is:

$$\Delta U_{a \rightarrow b} = U(r_b) - U(r_a) = - \int_{r_a}^{r_b} \vec{F} \cdot d\vec{l}$$

The negative sign occurs because it takes a positive amount of work to move an object *against* a force. For example, the stored potential gravitational energy increases by $mgh$ if you lift a mass $m$ upwards a distance $h$ against the downward gravitational force $mg$. 
Conservative Force

For a conservative force, the line integral

$$\Delta U_{a \to b} = U(r_b) - U(r_a) = -\int_{r_a}^{r_b} \mathbf{F} \cdot d\mathbf{l}$$

is *independent* of the path taken between the start and end points.

The gravitational and electric forces both are conservative.
Proof that the electric field is conservative; 1

Consider electric field produced by a point charge \( q_1 \). Evaluate the work done moving charge \( q_0 \) from \( a \) to \( b \) along paths A or B.

**Path A: Tangential plus radial paths**

From Coulomb’s law we have that the force of \( q_0 \) due to point charge \( q_1 \) is:

\[
\vec{F}_{10} = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_0}{r_{10}^2} \vec{r}_{10}
\]

Along path A we have that the change in potential energy is given by the line integral:

\[
\Delta U^A_{a\rightarrow b} = - \int_a^b \vec{F} \cdot d\vec{l} = \int_a^b \vec{F} \cdot d\vec{l}
\]

Note that along the tangent to a circle, from \( a \) to \( \alpha \), the integral is zero since the path is perpendicular to \( \vec{F} \) and thus the dot product \( \vec{F} \cdot d\vec{l} = 0 \). The radial integral is straightforward since the radial path is along \( \vec{F} \) and thus the integral is

\[
\Delta U^A_{a\rightarrow b} = \int_a^b \frac{q_1 q_0}{4\pi \varepsilon_0} \frac{\vec{r}}{r_{10}^2} \cdot \vec{r} dr
\]

\[
\Delta U^A_{a\rightarrow b} = \frac{q_1 q_0}{4\pi \varepsilon_0} \left[ \frac{1}{r_b} - \frac{1}{r_a} \right]
\]

This line integral just depends of the starting and ending positions.
Proof that the electric field is conservative;

Consider electric field produced by a point charge \( q_1 \). Evaluate the work done moving charge \( q_0 \) from \( a \) to \( b \) along paths A or B.

**Path B: Arbitrary path**

Path B can be factored into a sequence of infinitesimal radial and tangential steps as shown schematically. Now the line integrals along each of the tangential paths are zero because the path and force are perpendicular. Thus we are left with only the sum of a series of radial integrals.

\[
\Delta U^B_{a \to b} = - \sum \Delta r_i \int r_i + \Delta r_i \frac{q_1 q_0}{4 \pi \varepsilon_0 r_i^2} \hat{r} \cdot \hat{r} \, dr
\]

\[
\Delta U^B_{a \to b} = \frac{q_1 q_0}{4 \pi \varepsilon_0} \left[ \frac{1}{r_b} - \frac{1}{r_a} \right]
\]

because in the addition all the terms cancel except the first and last terms. Thus we have proven that the electric force is \textit{conservative} for two point charges.
Electric field for multiple charges

Now use the Principle of Superposition for an electric field produced by a set of $n$ charges.

$$\mathbf{F}_{net} = \sum_{i=1}^{n} \mathbf{F}_i$$

The line integral then can be written as:

$$\Delta U_{a \rightarrow b}^{net} = - \int_{r_a}^{r_b} \mathbf{F}_{net} \cdot d\mathbf{l}$$

$$= - \sum_{i=1}^{n} \int_{r_a}^{r_b} \mathbf{F}_i \cdot d\mathbf{l}$$

$$\Delta U_{a \rightarrow b}^{net} = \sum_{i=1}^{n} \Delta U_{a \rightarrow b}^{i}$$

Thus we see that the net potential energy difference is the sum of the contributions from each charge producing the electric field. Since each component is conservative, then the total potential energy difference also must be conservative.

Note that the electric force is conservative because the force is radial. Similarly, the proof that the gravitational force is conservative is identical except that the product $\frac{q_1 q_0}{4 \pi \varepsilon_0}$ is replaced by $-Gm_1 m_0$. 
Electric Potential

Using \( \mathbf{F} = q_0 \mathbf{E} \) gives that the change in potential energy due to moving a charge \( q_0 \) from \( a \) to \( b \) is:

\[
\Delta U_{a \to b}^{net} = -q_0 \int_{r_a}^{r_b} \mathbf{E}_{net} \cdot d\mathbf{l}
\]

Note that the probe charge \( q_0 \) factors out from the integral. Now we will define a new quantity called electric potential \( V \) where

\[
\Delta V_{a \to b}^{net} = \frac{\Delta U_{a \to b}^{net}}{q_0} = -\int_{r_a}^{r_b} \mathbf{E}_{net} \cdot d\mathbf{l}
\]

That is; electric potential difference is the work that must be done to move a unit charge from \( a \) to \( b \) with no change in kinetic energy. Be careful not to confuse electric potential energy difference \( \Delta U_{a \to b} \) and electric potential difference \( \Delta V_{a \to b} \), that is, \( \Delta U \) has units of energy, Joules, while \( \Delta V \) has units of Joules/Coulomb or volts. N.B. I have chosen to use the symbol \( V \) to designate electric potential, and \( \tau \) for volume to avoid confusion.
Electric Potential and Superposition

Previously it was shown that the electric force is conservative for the superposition of many charges. To recap, if \( \mathbf{E}_{\text{net}} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 \), then

\[
V_{a\to b}^{\text{net}} = -\int_{r_a}^{r_b} \mathbf{E}_{\text{net}} \cdot d\mathbf{l}
\]

\[
V_{a\to b}^{\text{net}} = -\int_{r_a}^{r_b} \mathbf{E}_1 \cdot d\mathbf{l} - \int_{r_a}^{r_b} \mathbf{E}_2 \cdot d\mathbf{l} - \int_{r_a}^{r_b} \mathbf{E}_3 \cdot d\mathbf{l}
\]

\[
V_{a\to b}^{\text{net}} = \sum_i^n V_{a\to b}^i
\]

Thus, like electric potential energy, electric potential is a simple additive scalar field because the Principle of Superposition applies.
**Electric Potential** \( V \)

**Units:**

- Electric potential: \( V \) = Volts = Joules/Coulomb
- Electric field: \( E \) = Volts/meter = Newton’s/Coulomb  \( [\mathbf{F} = q\mathbf{E}] \)

**Reference potentials:**

Only differences in potential energy \( U \) and electric potential \( V \) are significant

\[
V_{a\rightarrow b}^{net} = - \int_{r_a}^{r_b} \mathbf{E}_{net} \cdot d\mathbf{l}
\]

**Equipotentials:**

Surfaces of constant electric potential are perpendicular to the electric field \( \mathbf{E} \)
Electric Potential for Discrete Charges

Using Coulomb’s law it was shown that the change in electric potential energy from moving \( q_0 \) in the field of a single point charge \( q_1 \) from point \( a \) to \( b \) is:

\[
\Delta U_{a \rightarrow b} = \frac{q_1 q_0}{4\pi \varepsilon_0} \left[ \frac{1}{r_b} - \frac{1}{r_a} \right]
\]

Thus the change in electric potential due to the electric field from a point charge \( q_1 \) is:

\[
\Delta V_{a \rightarrow b} = \frac{q_1}{4\pi \varepsilon_0} \left[ \frac{1}{r_{b1}} - \frac{1}{r_{a1}} \right]
\]

where \( r_{a1} \) and \( r_{b1} \) are the distances of \( a \) and \( b \) from \( q_1 \). As shown before, the electric potential due to a system of \( n \) point charges is additive because of superposition, that is,

\[
\Delta V_{a \rightarrow b}^{net} = \sum_{i=1}^{n} \Delta V_{a \rightarrow b}^{i} = \sum_{i=1}^{n} \frac{q_i}{4\pi \varepsilon_0} \left[ \frac{1}{r_{bi}} - \frac{1}{r_{ai}} \right]
\]

If the reference, \( r_a \) is chosen at infinity, then one obtains that the electric potential at the point \( b \), for an ensemble of \( n \) fixed discrete charges, is

\[
\Delta V_{\infty \rightarrow b}^{net} = \sum_{i=1}^{n} \frac{q_i}{4\pi \varepsilon_0} \frac{1}{r_{bi}}
\]
Electric Potential for electric dipole
V for continuous charge distribution

Suppose charge is distributed over a volume $\tau$ with a density $\rho$ at any point within the volume. The electric potential at any field point $p$ due to an element of charge $dQ = \rho d\tau$ at a point $p'$ is given by:

$$\Delta V_{\infty \rightarrow p} = \int_{\tau} \frac{\rho(p')d\tau'}{4\pi \varepsilon_o r_{p'p}}$$

Again, the integral is over a scalar quantity.

Since electric potential $V$ is a scalar quantity, it is easier to compute than is the vector electric field $\vec{E}$. 
Consider a circular element, between $\theta$ and $\theta + \Delta \theta$ which will have a charge of

$$\Delta Q = 2\pi R \sin \theta R d\theta \sigma$$

At distance $r$ from the center of the shell, and outside of the shell, the potential from this circular element of charge, relative to $r = \infty$, is

$$\Delta V = \frac{\Delta Q}{4\pi \varepsilon_0 \beta} = \frac{2\pi R \sin \theta R d\theta \sigma}{4\pi \varepsilon_0 \beta}$$

where $\beta^2 = r^2 + R^2 - 2Rr \cos \theta$. The total potential is obtained by integrating over the whole sphere:

$$V = \int_0^\pi \frac{2\pi R \sin \theta R d\theta \sigma}{4\pi \varepsilon_0 \beta}$$

Since $2\beta d\beta = 2R \sin \theta d\theta$, then

$$V = \frac{\sigma R}{2\varepsilon_0 r} \int_{r-R}^{r+R} d\beta = \frac{\sigma R^2}{\varepsilon_0 r} = \frac{Q}{4\pi \varepsilon_0 r} \quad r > R$$

For $r$ inside the spherical charged shell the potential is:

$$V = \frac{\sigma R}{2\varepsilon_0 r} \int_{R-r}^{R+r} d\beta = \frac{\sigma R^2}{\varepsilon_0 R} = \frac{Q}{4\pi \varepsilon_0 R} \quad r < R$$

Note that outside of the spherical charge shell, the potential has a $\frac{1}{r}$ radial dependence, whereas it is a constant inside of the spherical shell.
Why determine the electric potential $V$?

- Use $V$ to determine the difference in electrostatic potential energy

$$U_{a \rightarrow b} = qV_{a \rightarrow b}$$

NB: Electron-volt unit of energy: $q=1.602 \times 10^{-19}$ C

$1 \text{ eV} \equiv 1.602 \times 10^{-19}$ Joules

- Use $V$ to determine the electric field $E$
Relation between equipotentials and force field
Determination of $E$ from $V$

The electric field and electric potential are directly related by:

\[ \Delta V_{a \rightarrow b} = - \int_{r_a}^{r_b} \vec{E} \cdot d\vec{l} \]

For an arbitrary infinitesimal element distance $d\vec{l}$ the change in electric potential $dV$ is

\[ dV = -\vec{E} \cdot d\vec{l} \]

Using cartesian coordinates both $\vec{E}$ and $d\vec{l}$ can be written as

\[ \vec{E} = \hat{i}E_x + \hat{j}E_y + \hat{k}E_z \]

\[ d\vec{l} = \hat{i}dx + \hat{j}dy + \hat{k}dz \]

Taking the scalar product gives:

\[ dV = -\vec{E} \cdot d\vec{l} = -E_x dx - E_y dy - E_z dz \]

Using differential calculus we know that one can express the change in potential $dV$ in terms of partial derivatives by:

\[ dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \]

By association, this gives:

\[ E_x = -\frac{\partial V}{\partial x}, \ E_y = -\frac{\partial V}{\partial y}, \ E_z = -\frac{\partial V}{\partial z} \]
Thus on each axis, the electric field is minus the gradient of the electric potential. In three dimensions, the electric field is minus the total gradient of electric potential. That is, the electric field is just the gradient of the electric potential, that is it is perpendicular to the equipotentials. Skiers are familiar with the concept of gravitational equipotentials and the fact that the line of steepest descent, and thus maximum acceleration, is perpendicular to gravitational equipotentials of constant height.

The gradient of the scalar function $V$ is written in mathematics as $\nabla V$ which is a vector quantity. Namely:

$$\vec{E} = -\nabla V$$

In cartesian coordinates this can be written as:

$$\vec{E} = - \left[ \hat{i} \frac{\partial V}{\partial x} + \hat{j} \frac{\partial V}{\partial y} + \hat{k} \frac{\partial V}{\partial z} \right]$$

Knowing the electric potential allows calculation of the electric field. The gradient of $V$, $\nabla V$, also can be expressed in spherical or cylindrical coordinates.
e.g. Uniformly charged spherical shell

It was shown that the equipotentials are concentric spheres and the electric potential is given by:

\[ V = \frac{\sigma R}{2\varepsilon_0 r} \int_{r-R}^{r+R} d\beta = \frac{\sigma R^2}{\varepsilon_0 r} = \frac{Q}{4\pi \varepsilon_0 r} \quad r > R \]

\[ V = \frac{\sigma R}{2\varepsilon_0 r} \int_{R-r}^{R+r} d\beta = \frac{\sigma R^2}{\varepsilon_0 R} = \frac{Q}{4\pi \varepsilon_0 R} \quad r < R \]

Taking the radial derivative one can calculate the \( E \) fields from this potential since

\[ \vec{E} = -\vec{\nabla}V = -\frac{dV}{dr} \hat{r} \]

gives that

\[ \vec{E}_{net} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \hat{r} \quad r > R \]

and

\[ \vec{E}_{net} = 0 \quad r < R \]

Gauss’ law gave the same answer.
Circulation of a static electric field

Since the electric field is conservative

\[ \Delta V_{a \to b}^{net} = \frac{\Delta U_{a \to b}^{net}}{q_0} = -\int_{r_a}^{r_b} \mathbf{E}_{net} \cdot d\mathbf{l} \]

is independent of the path taken between two points a and b. Consider two possible paths between a and b as shown. The line integral from a to b via route 1 is equal and opposite to the line integral back from b to a via route 2 if the electric field is conservative as shown earlier.

A better way of expressing this is that the line integral of \( \mathbf{E} \) is zero around any closed path. That is, the line integral between a and b, via path 1, and returning back to a, via 2, are equal and opposite. That is;

\[
\oint \mathbf{E}_{net} \cdot d\mathbf{l} = \int_{a}^{b} \mathbf{E}_{net} \cdot d\mathbf{l} + \int_{b}^{a} \mathbf{E}_{net} \cdot d\mathbf{l}
\]

\[= \int_{a}^{b} \mathbf{E}_{net} \cdot d\mathbf{l} - \int_{a}^{b} \mathbf{E}_{net} \cdot d\mathbf{l} = 0 \]

Therefore the net line integral for a closed loop is zero.

\[\oint \mathbf{E}_{net} \cdot d\mathbf{l} = 0 \]

This is a measure of the circulation of the electric field. The fact that it is zero is a statement that the electric field is radial for a point charge.
Summary

Electric potential

The concept of electric potential $V$ has been introduced where electric potential difference is given as:

$$\Delta V_{a \rightarrow b}^{\text{net}} = \frac{\Delta U_{a \rightarrow b}^{\text{net}}}{q_0} = - \int_{r_a}^{r_b} \mathbf{E}_{\text{net}} \cdot d\mathbf{l}$$

The derivation of $\mathbf{E}$ from $V$ was shown. In particular, $\mathbf{E}$ is given by the gradient of $V$, that is:

$$\mathbf{E} = -\nabla V$$

The application of electric potential to systems of electric conductors was discussed.

Maxwell’s equations for electrostatics

The laws of electrostatics have been rewritten in terms of the concepts of flux and circulation of a vector field. Gauss’s law gives the net flux for a closed surface:

$$\Phi_{\text{net}} = \oint_{\text{surface}} \mathbf{E}_{\text{net}} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\text{enclosed}} \rho d\tau$$

The flux relation tells you that the field for a point charge has $\frac{1}{r^2}$ dependence.

The net circulation for a closed loop:

$$\oint \mathbf{E}_{\text{net}} \cdot d\mathbf{l} = 0$$

The zero circulation is a statement that the electric field in Coulomb’s law is radial.