Summary

The magnetic field due to a moving charge is:

\[ \mathbf{B} = \frac{\mu_0}{4\pi} \frac{q \mathbf{v} \times \mathbf{r}}{r^2} \]

The Biot Savart Law gives the field at a point \( r \) from a circuit carrying current \( I \) as:

\[ \mathbf{B} = \frac{\mu_0}{4\pi} \oint_{\text{closed circuit}} \frac{I \mathbf{dl} \times \mathbf{r}}{r^2} \]

Discuss Ampère’s law and applications.

The magnetic force between two circuits is

\[ \mathbf{F}_a = -\frac{\mu_0}{4\pi} I_a I_b \oint_a \oint_b \frac{(\mathbf{dl}_a \cdot \mathbf{dl}_b)}{r_{ba}^2} \mathbf{r}_{ba} \]
• Magnetic dipole

• Microscopic magnetism in matter

• Macroscopic magnetism in matter

• Boundary conditions

• Electromagnetism
Comparison of electric and magnetic dipole fields

\[ \mathbf{p} = q \mathbf{d} \]

\[ \mathbf{\mu} = NIA\mathbf{n} \]
Microscopic magnetism in matter

As an example of typical atomic magnetic moments, consider a hydrogen atom. Equating the centripetal acceleration and the Coulomb force implies that the electron revolves around the proton with a velocity of $2.2 \times 10^6 \text{m/s}$ producing an average current of 1 milliamp in a circular orbit of radius $0.53 \times 10^{-10} \text{m}$. That is, the magnetic moment is $\mu = \pi r^2 I = 0.93 \times 10^{-23} \text{Am}^2$. In a model where all the atomic current loops are aligned, then the currents from adjacent atoms cancel in the interior leaving an effective current only at the surface of the material, as shown in figure 2.

That is, the material looks like a macroscopic-sized magnetic dipole due to the effective atomic surface current times the cross-sectional area of the material which can be $10^{16}$ times larger than the area of a single atom. This surface current, called an *amperian current*, is similar to the real current in a solenoid. This surface current produces an axial magnetic field in the material similar to that produced by a solenoid.
Microscopic magnetism in matter

It is useful to define the magnetization $\mathbf{M}$ which is the magnetic moment per unit volume.

$$\mathbf{M} = N \mathbf{\mu}$$

where there are $N$ magnetic dipoles per unit volume each of magnetic moment $\mathbf{\mu}$.

Consider Ampère’s law for magnetostatics

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\text{Enclosed current}} \mathbf{j} \cdot d\mathbf{S}$$

The current must include all possible currents, both free and bound currents, that pass through the surface bounded by the closed curve $C$. The only bound currents that contribute to the right-hand integral are those where the closed loop passes through the area prescribed by the circulating atomic current.
Microscopic magnetism in matter

As shown in figure 3, consider that there are $N$ atomic dipoles per unit volume that circulate in an orbit of radius $r$ and current $I$. Then for a path length $l$ there will be a net bound current around the line for all magnetic dipoles in the volume $\pi r^2 l \cos \theta$. That is

$$I_{\text{Bound}} = N \pi r^2 l \cos \theta = N \overrightarrow{\mu} \cdot \overrightarrow{l} = \overrightarrow{M} \cdot \overrightarrow{l}$$

which is independent of the size and other details of the atomic magnetic moments. Thus for a closed loop, the net bound current passing through the enclosed loop is

$$I_{\text{Bound}} = \oint \overrightarrow{M} \cdot d\overrightarrow{l}$$

Figure 3 Bound current due to atomic electron orbits of radius $r$ that are intercepted by the line $C$ at an angle $\theta$ to the magnetic dipoles.
**Microscopic magnetism in matter**

Inserting this bound current into Ampère’s law for magnetostatics gives

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint_{\text{Enclosed}} \mathbf{j}_{\text{free}} \cdot d\mathbf{S} + \mu_0 \oint \mathbf{M} \cdot d\mathbf{l} \]

where \( \mathbf{j}_{\text{free}} \) is the applied current density.

In the absence of magnetic material, the magnetic field, called \( \mathbf{B}_{\text{applied}} \), is given by

\[ \oint_C \mathbf{B}_{\text{applied}} \cdot d\mathbf{l} = \mu_0 \oint_{\text{Enclosed}} \mathbf{j}_{\text{free}} \cdot d\mathbf{S} \]

It is useful to define a magnetic susceptibility given by the relation

\[ \mathbf{M} = \chi_m \left( \frac{\mathbf{B}_{\text{applied}}}{\mu_0} \right) \]

Inserting this into Ampère’s Law gives that

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \oint_C \mathbf{B}_{\text{applied}} \cdot d\mathbf{l} + \chi_m \oint_C \mathbf{B}_{\text{applied}} \cdot d\mathbf{l} \]

That is, since the line integral \( C \) is the same, the actual magnetic field in matter is given by

\[ \mathbf{B} = (1 + \chi_m) \mathbf{B}_{\text{applied}} \]
Microscopic magnetism in matter

Define the relative permeability $\kappa_m$

$$\kappa_m = 1 + \chi_m$$

Then this gives that the actual magnetic field in matter is related to the applied magnetic field by

$$\overrightarrow{B} = \kappa_m \overrightarrow{B_{\text{applied}}}$$

In the absence of magnetic material, the magnetic field, called $\overrightarrow{B_{\text{applied}}}$, is given by

$$\oint_C \overrightarrow{B_{\text{applied}}} \cdot d\overrightarrow{l} = \mu_0 \int_{\text{Enclosed current}} \overrightarrow{j_{\text{free}}} \cdot d\overrightarrow{S}$$

Ampère’s Law in the presence of matter can be rewritten using the relation between $\overrightarrow{B}$ and $\overrightarrow{B_{\text{applied}}}$:

$$\oint_C \frac{1}{\kappa_m} \overrightarrow{B} \cdot d\overrightarrow{l} = \mu_0 \int_{\text{Enclosed current}} \overrightarrow{j_{\text{free}}} \cdot d\overrightarrow{S}$$

This relation is very important for solving magnetic circuits as will be illustrated later. Note that this implies a much stronger $B$ field for a given enclosed current in the case of a magnetic material with a large relative permeability.
Ampère’s Law in the presence of matter can be rewritten using the relation between $B$ and $B_{\text{applied}}$:

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The magnetic properties of matter have been buried into the relative permeability $\kappa_m$

This is analogous to the dielectric constant with the notable difference that the actual magnetic field inside matter is $\kappa_m$ times the field in the absence of matter, whereas, the actual electric field inside matter is reduced by the factor $\kappa_E$, that is:

$$\mathbf{B}_{\text{matter}} = \kappa_m \mathbf{B}_{\text{applied}}$$

$$\mathbf{E}_{\text{matter}} = \frac{\mathbf{E}_{\text{applied}}}{\kappa_E}$$
### Three flavors of magnetism in matter

<table>
<thead>
<tr>
<th>Material</th>
<th>Susceptibility, $\chi_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Diamagnetic</strong></td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td>$-9.1 \times 10^{-6}$</td>
</tr>
<tr>
<td>Copper</td>
<td>$-9.6 \times 10^{-6}$</td>
</tr>
<tr>
<td>Silver</td>
<td>$-2.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>Carbon</td>
<td>$-2.2 \times 10^{-5}$</td>
</tr>
<tr>
<td>(diamond form)</td>
<td></td>
</tr>
<tr>
<td>Bismuth</td>
<td>$-1.7 \times 10^{-4}$</td>
</tr>
<tr>
<td><strong>Paramagnetic</strong></td>
<td></td>
</tr>
<tr>
<td>Sodium</td>
<td>$7.2 \times 10^{-6}$</td>
</tr>
<tr>
<td>Cupric oxide</td>
<td>$2.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>Aluminum</td>
<td>$2.2 \times 10^{-5}$</td>
</tr>
<tr>
<td>Liquid oxygen (90 K)</td>
<td>$3.5 \times 10^{-3}$</td>
</tr>
<tr>
<td><strong>Ferromagnetic</strong></td>
<td></td>
</tr>
<tr>
<td>Iron (annealed)</td>
<td>$5.5 \times 10^3$</td>
</tr>
<tr>
<td>Permalloy (55% Fe, 45% Ni)</td>
<td>$2.5 \times 10^4$</td>
</tr>
<tr>
<td>Mu-metal (77% Ni, 16% Fe, 5% Cu, 2% Cr)</td>
<td>$1 \times 10^5$</td>
</tr>
</tbody>
</table>
Diamagnetism and Paramagnetism

Diamagnetism:
Magnetic induction, Faraday’s law, induces electron currents to try to oppose any applied magnetic field. Always exists but is negligibly small effect.

Paramagnetism:
Atomic magnetic moments due to unpaired electrons orbits try to align with applied magnetic field slightly enhancing the $B$ field.
Ferromagnetism

• For selected elements like Fe, Co, Ni, Gd, Sm, Dy, the atomic magnetic moments are large enough to cause strong magnetic interactions between neighboring atoms resulting in small regions where the magnetic moments all are aligned.

• The regions of magnetic alignment called magnetic domains are $\sim 10^{-2}\text{mm}$ in size.
Ferromagnetism

• Saturation of magnetization occurs

• Hysteresis occurs for $\mathbf{B}$ versus $\mathbf{B}_{\text{applied}}$

• Ferromagnetism disappears above the Curie temperature
Magnetic materials summary

- **Diamagnetism:** $\kappa_m \sim 0.99999$

- **Paramagnetism:** $\kappa_m \sim 1.0001$

- **Ferromagnetism:** $\kappa_m$ ranges from 1000 to 100,000

Magnetization in matter only significant for ferromagnetic materials
Boundary Conditions

- The lines of magnetic field are continuous. Therefore the normal component of $\mathbf{B}$ is continuous.

- If there are no free electric currents at the surface then

$$\oint_{C} \frac{\mathbf{B}}{\kappa_{m}} \cdot d\mathbf{l} = 0$$

This implies that the parallel component of $\mathbf{B}/\kappa_{m}$ is continuous across the boundary. That is, Amperian currents on the surface enhance the $\mathbf{B}$ field inside matter by the factor $\kappa_{m}$. 

Iron core increases magnetic field by a large factor.
Boundary Conditions

Figure 7  (a) The tangential and normal component of the electric field at a boundary of a dielectric. (b) The tangential and normal component of magnetic field at the boundary of magnetic material.
Toroidal solenoid wrapped on ferromagnetic core

To illustrate the impact of a ferromagnetic material on the design of magnets and transformers, consider for simplicity the N turn toroidal solenoid shown in figure 8. In the absence of matter inside the toroidal core, the B field at a radius r inside the windings of the toroid, when carrying a current I, is given by Ampère’s Law expressed as:

$$\oint \frac{1}{\kappa_m} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint_{Enclosed_{current}} \mathbf{j}_{free} \cdot d\mathbf{S}$$

That is, since \( \kappa_m = 1 \), then:

$$B = \mu_0 \frac{NI}{2\pi r}$$

Note that the maximum B field is obtained by making \( NI \) large and the circumference \( l = 2\pi r \) small.

Filling the inside of the toroid with a ferromagnetic material with a large relative permeability \( \kappa_m \) the magnetic field inside the toroid is given by:

$$B = \mu_0 \kappa_m \frac{NI}{2\pi r}$$

This is enhanced by the relative permeability which can be 5500 for iron.
Toroidal solenoid wrapped on ferromagnetic core

The ferromagnetic core of the toroid maintains a very large B field inside the toroid even if the N turn winding is not uniform. The ferromagnetic material acts like the path of least resistance for the magnetic field. Thus the windings can be concentrated in a region of the core with little influence on the internal magnetic field. Consequently, a transformer is built by winding a primary coil and separate secondary coil both wrapped on a ferromagnetic core made of minimum hysteresis iron, as shown in figure 9. The core usually is laminated to reduce eddy current losses due to the alternating magnetic field.
An electromagnet comprises a ferromagnetic core with a gap. Electromagnets are used extensively in science and technology. Consider the electromagnet formed by cutting a gap in the ferromagnetic core as shown in figure 10. Assuming that the applied magnetic field follows the path of the iron, then from Ampère’s Law:

$$\oint_{C} \frac{1}{\kappa_m} \vec{B} \cdot d\vec{l} = \mu_0 \int_{Enclosed \ current} \vec{j}_{\text{free}} \cdot d\vec{S}$$

Then

$$\oint \frac{1}{\kappa_m} \vec{B} \cdot d\vec{l} = \frac{1}{\kappa_m} B_{\text{iron}} L_{\text{iron}} + B_{\text{gap}} L_{\text{gap}} = \mu_0 NI$$

Now we know from the boundary condition that the normal component of $\vec{B}$ is continuous, that is it is unchanged inside and outside the iron, then,

$$B_{\text{iron}} = B_{\text{gap}}$$

Therefore

$$B_{\text{gap}} = \frac{\mu_0 NI}{(L_{\text{gap}} + \frac{L_{\text{iron}}}{\kappa_m})}$$

Since the iron core has relative permeability of 5500, then the contribution to the line integral for the magnetic path length in the iron is negligible. Thus:

$$B_{\text{gap}} \approx \frac{\mu_0 NI}{L_{\text{gap}}}$$
Iron saturates at an internal magnetic field of $B_{sat} = \mu_0 M = 2.16T$ (21.6 kgauss), thus typical conventional electromagnets are useful up to about 20kG. For stronger magnetic fields it is necessary to rely on very large ampere-turn values, iron is used only for the return yoke to stop the magnetic field from permeating large distances. A practical magnet usually has a larger cross sectional area for the iron yoke, than the area of the air gap, to spread out the magnetic flux which minimizes saturation effects in the iron. Figure 11 shows a typical electromagnet, where $I = 1\text{amp}$, $N = 10,000$, $A_{air} = 50cm^2$, $A_{iron} = 100cm^2$, then $B_{air} = 1.3T = 13kgauss$. If the iron core was removed then the magnetic field in the air gap would drop to about 800gauss.
Summary

• Ferromagnetic materials greatly enhance magnetic fields and have important technical applications.

• Diamagnetism and paramagnetism have negligible effects on the magnetic field and have few practical applications.

• The magnetic effects can be absorbed in the relative permeability $\kappa_m$.

• For magnetic materials use the modified form of Ampère’s Law

$$\oint_C \frac{1}{\kappa_m} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\text{Enclosed current}} \mathbf{j}_{\text{free}} \cdot d\mathbf{S}$$