Faraday’s Law

Faraday’s important contribution was his discovery that a changing magnetic flux induces an emf in a circuit. His relation is given as:

$$\xi = -\frac{d\Phi}{dt}$$

where the electromotive force $\xi$ is given by:

$$\xi = \int_{\text{Closed circuit}} (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$

and the magnetic flux $\Phi$ is given by:

$$\Phi = \int_{\text{Surface bounded by } C} \vec{B} \cdot d\vec{S}$$

Independent of whether the circuit moves or not, Faraday’s law is equivalent to the statement that:

$$\int_{\text{Closed circuit}} \vec{E} \cdot d\vec{l} = -\int_{\text{Surface bounded by } C} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

Remember that the line integral and surface integrals must be related by the right-hand rule.
AC Circuits

- AC circuits are of crucial importance for power distribution and circuit design.

- For a sinusoidal voltage applied to a circuit both the amplitude and phase of the current output waveform are influenced by the circuit.

- The voltage and current waveforms across a resistor are in phase.

- The voltage across a capacitor lags behind the current by 90° since charge is required to flow into the capacitor to change the voltage.

- The voltage across an inductor leads the current by 90° because of the inertia in build up of current in the inductor.

- The use of complex numbers facilitates dealing with AC circuits. Because of the mathematical complexity AC circuits will not be discussed further in P114. Read Giancoli chapter 31 if interested.
Magnetic Energy Stored in an Inductor

Since forces occur between magnetic circuits, energy must be stored in the magnetic field. Consider the system shown in figure 9.

Using Kirchhoff’s loop rule we have:

\[ \xi_0 = IR + L \frac{dI}{dt} \]

Consider that in a time \( dt \), a charge \( dQ = Idt \) flows. The work done by the battery is given by \( \xi_0 dQ \). This equals:

\[ \xi_0 Idt = I^2 R dt + LI \frac{dI}{dt} dt \]

This is equivalent to the statement that the energy provided by the battery equals the energy dissipated in the resistor plus the energy stored in the self inductance. Thus we have that the energy stored in the inductance is:

\[ dU = LI dI \]

Integrating the energy from \( I = 0 \) to the final value gives the magnetic energy stored in the self inductance as:

\[ U = \int_0^I LI dI = \frac{1}{2} LI^2 \]
Consider a simple LC circuit, shown in figure 10, with oscillating current and charge. The total energy is distributed between the capacitor and the inductor as:

\[ U_{Tot} = U_{elec} + U_{mag} \]

\[ U_{Tot} = \frac{1}{2} \frac{Q^2}{C} + \frac{1}{2} LI^2 \]

Note that the energy oscillates between the capacitor, when \( Q \) is maximum and \( I = 0 \), to the inductor when \( Q = 0 \) and \( I \) is a maximum. This is analogous to harmonic oscillations of a pendulum where the energy oscillates between kinetic energy and potential energy. The inertia in the inductance is analogous to moment of inertia in the kinetic energy term for angular motion of the pendulum. The energy stored in the capacitor is analogous to the gravitational potential energy stored at the extreme positions of the pendulum oscillation.
Energy density in a magnetic field

It is more useful to express the stored magnetic energy density in terms of the magnetic field $B$, just as the electric energy density was expressed in terms of the electric field $E$. In the case of the electric field, the stored electric energy for a capacitor, of $U_E = \frac{1}{2} CV^2$ was used to show the the electric energy can be expressed as the integral of the electric energy density $\eta_E$ in vacuum

$$\eta_E = \frac{1}{2} \varepsilon_0 \kappa_e E^2$$

Thus the total stored energy in the electric field in vacuum

$$U_E = \int_{All \ space} \frac{1}{2} \varepsilon_0 \kappa_e E^2 d\tau$$

where the integral is taken over all space.

For the magnetic field it can be shown that the magnetic energy $U_{mag} = \frac{1}{2} LI^2$ can be expressed in terms of the magnetic energy density $\eta_B$ in vacuum

$$\eta_B = \frac{B^2}{2\mu_0 \kappa_m}$$

Thus the total stored energy in the magnetic field in vacuum is

$$U_B = \int_{All \ space} \frac{1}{2} \frac{B^2}{\mu_0 \kappa_m} d\tau$$
Self inductance and stored energy for a toroid

Figure 14  N turn toroid with inner radius $a$, outer radius $b$, and thickness $l$. 
Total electromagnetic energy density

It is especially useful to express the total energy stored in an electromagnetic field in terms the energy density of the E and B fields.

\[ U_{Total} = \int_{all\ space} \left( \frac{1}{2} \varepsilon_0 \kappa_e E^2 + \frac{1}{2} \frac{B^2}{\mu_0 \kappa_m} \right) d\tau \]

This form will be used in discussing electromagnetic radiation.
Summary

The concepts of self inductance and mutual inductance and some applications have been discussed. It was shown that the induced emf in an isolated circuit can be written as:

$$\xi_a = -L_a \frac{dI_a}{dt}$$

where the self inductance $L_a$ is defined as:

$$L_a \equiv \frac{d\Phi_{aa}}{dI_a}$$

Similarly, the coupling between two circuits is given by

$$\xi_a = -M_{ab} \frac{dI_b}{dt}$$

where the mutual inductance $M_{ab}$ is defined as:

$$M_{ab} \equiv \frac{d\Phi_{ab}}{dI_b}$$

Elements of the response of circuits with $L$, $R$, and $C$ were discussed as well as applications to the transformer and the induction coil.

It was shown that the energy stored in an inductor is given by

$$U_B = \frac{1}{2}LI^2$$

Thus the total energy stored in a circuit having both inductors and capacitors is:

$$U_{Tot} = U_{elec} + U_{mag}$$

$$U_{Tot} = \frac{1}{2}Q^2 / C + \frac{1}{2}LI^2$$

It is especially useful to express the total energy stored in an electromagnetic field in terms the energy density of the $E$ and $B$ fields.

$$U_{Total} = \int_{all \ space} \left( \frac{1}{2}\varepsilon_0 \kappa_e E^2 + \frac{1}{2} \mu_0 \kappa_m B^2 \right) d\tau$$
- Electrodynamics before Maxwell
- Maxwell’s Displacement current
- Maxwell’s Equation’s
- Maxwell’s Equations in vacuum
- Mathematics of waves
Electrodynamics before Maxwell

**Flux:** Gauss’s Law:
Coulomb’s Law implies that the electric flux out of a closed surface is:

$$
\Phi_E = \oint_{\text{Closed surface}} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int_{\text{enclosed volume}} \rho d\tau
$$

The Biot Savart Law implies that the magnetic flux out of a closed surface is:

$$
\Phi_B = \oint_{\text{Closed surface}} \vec{B} \cdot d\vec{S} = 0
$$

Although these two flux equations were derived only for statics, they also are obeyed for electrodynamics.

**Circulation:**
Faraday’s Law implies that the circulation of the electric field is coupled to the rate of change of magnetic flux.

$$
\oint_{\text{Closed circuit}} \vec{E} \cdot d\vec{l} = -\int_{\text{Surface bounded by } C} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}
$$

Ampère’s law gives the circulation of the magnetic field:

$$
\oint_{\text{Closed loop } C} \vec{B} \cdot d\vec{l} = \mu_0 \oint_{\text{Bounded by } C} \vec{j} \cdot d\vec{S}
$$
Charge conservation

Charge conservation is an important law of nature that must be obeyed by the laws of electrodynamics. Charge conservation implies that the net current flowing out of a closed surface must equal the rate of loss of charge from the enclosed volume.

\[ \oint_{\text{Closed surface}} \mathbf{j} \cdot d\mathbf{S} + \int_{\text{Enclosed volume}} \frac{\partial \rho}{\partial t} \, dV = 0 \]

Maxwell realized that the above four equations for electrodynamics violate charge conservation. He showed that an additional term, called the displacement current, that depends on the rate of change of electric field, missing from Ampère’s Law; is needed to satisfy charge conservation. This lecture will first discuss the need for the introduction of a new displacement current term to satisfy charge conservation. This will be followed by a summary of the fundamental Maxwell Equations.
Maxwell’s Displacement Current

Maxwell showed that Ampère’s law could be generalized and made to satisfy charge conservation, by including an additional term to the current density called the displacement current density, \( \vec{j}_D \), where:

\[
\vec{j}_D \equiv \varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\]

Addition of this displacement current density to the real current density \( \vec{j} \) gives the Maxwell-Ampère law:

\[
\int_C \vec{B} \cdot d\vec{l} = \mu_0 \int_{\text{Surface bounded by } C} \vec{j} \cdot d\vec{S} + \mu_0 \varepsilon_0 \int_{\text{Surface bounded by } C} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{S}
\]

The second integral equals \( \mu_0 \int \vec{j}_D \cdot d\vec{S} \) that is the Maxwell-Ampère includes the sum of the the real current density and the displacement current.
Proof of need for displacement current

A formal proof for the need of the displacement current term is as follows. Consider that the closed loop \( C \), in figure 1, shrinks to zero, that is \( C \to 0 \), then the surface integral becomes a closed surface. That is:

\[
\oint_{C=0} \mathbf{B} \cdot d\mathbf{l} = 0 = \mu \oint \mathbf{j} \cdot d\mathbf{S} + \mu_0 \varepsilon_0 \oint \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}
\]

Gauss’ law gives:

\[
\oint_{\text{Closed surface}} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\text{Enclosed volume}} \rho d\tau
\]

therefore, the time derivative of Gauss’s law:

\[
\oint_{\text{Closed surface}} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\text{Enclosed volume}} \frac{\partial \rho}{\partial t} d\tau
\]

Inserting this into the above line integral for a closed loop \( C \) gives

\[
\oint_{\text{Closed surface}} \mathbf{j} \cdot d\mathbf{S} + \int_{\text{Enclosed volume}} \frac{\partial \rho}{\partial t} d\tau = 0
\]

that is, the charge conservation relation is obtained. Inclusion of the displacement current term leads to the second integral which is essential to this proof.
Charging a capacitor 1

An understanding of Maxwell’s displacement current density can be obtained by considering the case of a capacitor charging with current $I$ as illustrated in figure 2. The current in the wire produces a measurable magnetic field circling around the wire carrying the current $I$.

For surface $S_1$, bounded by the closed loop $C$, there is a real current $I$ passing inwards through this surface which is related to the non-zero circulation of $B$ around the closed loop $C$ given by Ampère’s law. However, as shown in figure 2, surface $S_2$, which passes between the capacitor plates, has no real current flowing through its surface, and thus Ampère’s law implies that there is no magnetic field around $C$. However, there remains a measurable magnetic field around the wire carrying the current $I$. Clearly something is wrong with Ampère’s Law.
Maxwell’s displacement current corrects this flaw in Ampère’s Law, and satisfies charge conservation. Inclusion of the displacement current does not change the integral of current though surface $S_1$ since the displacement current is zero for this surface because $\varepsilon_0 \int_{S_1} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} = 0$, whereas the real current is $I$. However, for $S_2$, the real current density is zero but the displacement current is non-zero since the electric field between the capacitor plates in increasing. It is shown below that the net current through both $S_1$ and $S_2$ are the same if both the real and displacement currents are summed.
Charging capacitor 3

The left-hand integral has a zero contribution for surface $S_1$. Thus the net displacement current is given by:

$$\int_{S_1} \overrightarrow{j_D} \cdot d\overrightarrow{S} = \varepsilon_0 \int_{S_2} \frac{\partial \overrightarrow{E}}{\partial t} \cdot d\overrightarrow{S}$$

Inserting this in the charge conservation relation

$$\oint_{Closed\ surface} \overrightarrow{j} \cdot d\overrightarrow{S} + \oint_{Enclosed\ volume} \frac{\partial \rho}{\partial t} d\tau = 0$$

and using the fact that the real current density is non-zero only for $S_1$, and displacement current is non-zero only for $S_2$ gives

$$\int_{S_1} \overrightarrow{j} \cdot d\overrightarrow{S} + \int_{S_2} \overrightarrow{j_D} \cdot d\overrightarrow{S} = 0$$

$$\int_{S_1} \overrightarrow{j} \cdot d\overrightarrow{S} = \int_{S_2} \overrightarrow{j_D} \cdot d\overrightarrow{S} = I$$

where the direction of the first integral has been reversed. That is, the total displacement current flowing through $S_2$ equals the total real current flowing through $S_1$. 
Maxwell-Ampère Law

Thus the Maxwell-Ampère’s law can be written:

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{S_1} \mathbf{j} \cdot d\mathbf{S} + \mu_0 \varepsilon_0 \int_{S_2} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} \]

where the first integral equals \( \mu_0 I \) for surface \( S_1 \) and zero for surface \( S_2 \), while the second integral equals \( \mu_0 I \) for surface \( S_2 \) and is zero for surface \( S_1 \). That is, the circulation of \( \mathbf{B} \) around the loop \( C \) is the same whether surface \( S_1 \) or surface \( S_2 \) are used. Thus Maxwell’s corrected version of Ampère’s Law satisfies charge conservation, that is, the circulation of magnetic field is independent on whether a current is flowing or the electric flux is changing, that is whether surface \( S_1 \) or surface \( S_2 \) are used for the closed loop \( C \).
Magnetic field between plates of charging parallel-plate capacitor

Use symmetry plus the fact that $\mathbf{j}$ is zero between the plates

$$\oint C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{j} \cdot d\mathbf{S} + \mu_0 \varepsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

$$B 2\pi r = \mu_0 0 + \mu_0 \varepsilon_0 \pi r^2 \frac{dE}{dt}$$

Using Gauss’s Law at the surface of the capacitor plates gives that

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 \pi R^2}$$

These give that

$$B = \frac{\mu_0 r}{2\pi R^2} \frac{dQ}{dt} = \frac{\mu_0 r}{2\pi R^2} I$$

This result is the same as for the magnetic field inside a cylindrical conductor of radius $R$ having a uniform current density $\frac{I}{\pi R^2}$. However, this result was obtained by calculating the induced magnetic field due to a changing flux of electric field.
Maxwell’s Equations

Flux
Gauss’s Law for electric field:

\[ \Phi_E = \oint_{\text{Closed surface}} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\text{enclosed volume}} \rho d\tau \]

= \frac{1}{\varepsilon_0} (Enclosed charge)

Gauss’s law for magnetism

\[ \Phi_B = \oint_{\text{Closed surface}} \mathbf{B} \cdot d\mathbf{S} = 0 \]

Circulation:
Faraday’s Law

\[ \oint_{\text{Closed loop}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\text{surface bounded by } C} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \]

Ampère-Maxwell law:

\[ \oint_{\text{Closed loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\text{Bounded by } C} \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{S} \]

= \mu_0 (Net real and displacement currents flowing through the loop)

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]
Maxwell’s Equations in Vacuum

\[ \mathbf{j} = \rho = 0. \]

\[ \oint \mathbf{E} \cdot d\mathbf{S} = 0 \quad \text{(Closed surface)} \]

\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\int_{\text{surface bounded by } C} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \]

\[ \oint \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{(Closed surface)} \]

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} \quad \text{(Bounded by } C) \]

- Note the symmetry of these four equations. Only the sign and the product \( \mu_0 \varepsilon_0 \) are nonsymmetric.

- The product \( \mu_0 \varepsilon_0 = 1/c^2 \) where \( c \), the velocity of light in vacuum, is a fundamental constant of nature.
Next lecture will show and demonstrate that Maxwell’s Equations in vacuum predict the existence of electromagnetic waves that travel with the velocity of light $c$, where the product $\mu_0\varepsilon_0 = 1/c^2$ is a fundamental constant of nature.

The discussion of electromagnetic waves requires knowledge of the Wave Equation which will be introduced next.
Mathematics of Waves

Consider a travelling wave in one dimension. If the wave is moving, then the wave function $\Psi(x,t)$ describing the shape of the wave, is a function of both $x$ and $t$. The instantaneous amplitude of the wave $\Psi(x,t)$ could correspond to the transverse displacement of a wave on a string, the longitudinal amplitude of a wave on a spring, the pressure of a longitudinal sound wave, the electric or magnetic fields in an electromagnetic wave, a matter wave, etc. If the wave train maintains its shape as it moves, then one can describe the wave train by the function $f(\phi)$ where the coordinate $\phi$ is measured relative to the shape of the wave, that is, it is like a phase. Consider that $f(\phi = 0)$, corresponds to the peak of the travelling pulse shown in figure 4. If the wave travels at velocity $v$ in the $x$ direction, then the peak is at $x = 0$ for $t = 0$, and is at $x = vt$ at time $t$. That is: the fixed point on the wave profile $f(\phi)$ moves in the following way

$$\phi = x - vt \quad \text{moving in } +x \text{ direction}$$

$$\phi = x + vt \quad \text{moving in } -x \text{ direction}$$

That is, any arbitrary shaped wave form travelling in either $x$ direction can be written as:

$$\Psi(x,t) = f(x \pm vt) = f(\phi)$$

where the phase $\phi$ is fixed with respect to the waveform shape. It can correspond to motion in either direction.
Wave Equation

General wave motion can be described by solutions of a wave equation. The wave equation can be written in terms of the spatial and temporal derivatives of the wave function $\Psi(x,t)$. Consider the first partial derivatives of $\Psi(x,t) = f(x \mp vt) = f(\phi)$.

$$\frac{\partial \Psi}{\partial x} = \frac{d \Psi}{d \phi} \frac{\partial \phi}{\partial x} = \frac{d \Psi}{d \phi}$$

and

$$\frac{\partial \Psi}{\partial t} = \frac{d \Psi}{d \phi} \frac{\partial \phi}{\partial t} = \mp v \frac{d \Psi}{d \phi}$$

Factoring out $\frac{d \Psi}{d \phi}$ for the first derivatives gives

$$\frac{\partial \Psi}{\partial t} = \mp v \frac{\partial \Psi}{\partial x}$$

This wave equation in one dimension is independent of the sign of the velocity. There are an infinite number of possible shapes of waves travelling in one dimension, all of these must satisfy this one-dimensional wave equation. The converse is that any function that satisfies this one dimensional wave equation must be a wave in this one dimension.
Wave Equation

One example of a solution of this one-dimensional wave equations is the sinusoidal wave

\[ \Psi(x,t) = A \sin(\omega [t - \frac{x}{v}]) = A \sin(\omega t - kx) \]

where \( v = \frac{\omega}{k} \). The wave number \( k = \frac{2\pi}{\lambda} \), where \( \lambda \) is the wavelength of the wave, and angular frequency \( \omega = 2\pi v \). Note that this satisfies the above wave equation where the wave velocity equals \( v = v \lambda = \frac{\omega}{k} \).

The Wave Equation in three dimensions is

\[ \nabla^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \]

There are an infinite number of possible solutions \( \Psi \) to this wave equation, any one of which corresponds to a wave motion with velocity \( v \).

The Wave Equation is applicable to all forms of wave motion, both transverse and longitudinal. That is, it applies to waves on a string, seismic waves, water waves, sound waves, electromagnetic waves, matter waves, etc. In the subsequent discussion of electromagnetic waves, the Maxwell Equations will be used to deduce a wave equation. The existence of a wave equation is equivalent to proving the existence of electromagnetic waves of any wave form, frequency or wavelength travelling with the velocity given by the wave equation.
Maxwell’s Equations

**Flux**

Gauss’s Law for electric field:

\[ \Phi_E = \oint_{\text{Closed surface}} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{\text{enclosed volume}} \rho d\tau \]

\[ = \frac{1}{\varepsilon_0} (\text{Enclosed charge}) \]

Gauss’s law for magnetism

\[ \Phi_B = \oint_{\text{Closed surface}} \mathbf{B} \cdot d\mathbf{S} = 0 \]

**Circulation:**

Faraday’s Law

\[ \oint_{\text{Closed loop}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\text{surface bounded by } C} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \]

Ampère-Maxwell law:

\[ \oint_{\text{Closed loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint_{\text{Bounded by } C} \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{S} \]

\[ = \mu_0 (\text{Net real and displacement currents flowing through the loop}) \]

\[ \mathbf{F} = q(\mathbf{E} + \nabla \times \mathbf{B}) \]
Summary

The Maxwell equations, which are the fundamental laws of electromagnetism, have been obtained. They play a crucial role in most branches of science and engineering. The Maxwell Equations apply to all manifestations of electromagnetism and they contain Einstein’s Theory of Special Relativity. For example Maxwell’s equations predict that the velocity of light is independent of motion of the frame of reference. Maxwell did not realize this important facet of his equations: it took Einstein to discover this crucial fact which he presented in his seminal 1905 paper entitled ”On the electrodynamics of moving bodies”.