Chapter 7  ELECTRIC CURRENT

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INTRODUCTION

The first six lectures have focussed on electrostatics. Coulomb’s Law plus the Principle of Superposition, both of which are experimental facts, led to corresponding relations for flux and circulation of the vector electrostatic field; namely;

\[ \Phi_{net} = \oint_{surface} \mathbf{E}_{net} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_{enclosed} \rho \, d\tau \]

\[ \oint \mathbf{E}_{net} \cdot d\mathbf{l} = 0 \]

The flux relation tells you that the field for a point charge has \( \frac{1}{r^2} \) dependence. The zero circulation is a statement that the electric field in Coulomb’s law is radial. The concepts of flux and circulation provide a general representation of Coulomb’s law that is applicable to both static and non static electric fields. You will see later that, for changing electromagnetic fields, the circulation equation is modified with crucially important consequences.

These laws led to introduction of the concept of electric potential, \( V \), and the derivation of electric field from electric potential.

\[ \mathbf{E} = -\nabla V \]

This was applied to static electric fields and potentials around electric conductors leading to capacitance and the impact of dielectrics. The final discussion explored the use of electrostatic energy.

The course will now introduce the dynamics of moving charges, that is electric currents.

ELECTRIC CURRENT

An electric current will flow if free charges occur in an electric field. Electric current is the total charge that passes through a given cross section area per unit time.

\[ I = \frac{dq}{dt} \]

The unit of current is the ampere = Coulomb/second. This is a convenient unit, typical home appliances draw currents of a few amperes. However, currents can range from \( 10^{-15} \)amp in sensitive circuits to \( 10^7 \)amp in superconductors.

It is useful to define a current density \( \mathbf{j} \) which is the rate of charge flow per unit area through an infinitesimal area. Note that the current density is a vector since current flow has a direction and magnitude. The current \( I \) flowing through a surface \( A \) is given by the integral of the normal component of the current density to the surface area;

\[ I = \int_{surface} \mathbf{j} \cdot d\mathbf{S} = \int_{surface} j \, dS \cos \theta \]

Consider a collection of \( n \) charges per unit volume, each of charge \( q \) and moving at a drift velocity \( \mathbf{v} \). The number density, \( n \), times the charge, \( q \), gives the charge density. The charge \( \Delta Q \) that flows through an area \( A \) in time \( \Delta t \) is given by the charge density, \( nq \), and the volume of length \( v \Delta t \) and cross sectional area \( A \).

\[ \Delta Q = (nq)(A v \Delta t) = n q v A \Delta t = I \Delta t \]

That is:

\[ \mathbf{I} = nq \mathbf{v} A \]

The current density then is given by \( \frac{\mathbf{I}}{A} \), i.e.:

\[ \mathbf{j} = nq \mathbf{v} \]

That is, the current density is the charge density times the drift velocity. In matter, there can be several types of charge carriers, with number density \( n_i \) and charge
Figure 2  Current flowing out of a closed surface containing enclosed charge.

$q_i$, each with their own drift velocity $v_i$. The net current density then is given by:

$$\mathbf{j} = \sum_i n_i q_i \mathbf{v}_i$$

Note that the current density does not differentiate between positive charges moving in the $+\mathbf{v}$ direction or negative charges moving in the $-\mathbf{v}$ direction. In a metal, the free charges are negatively-charged electrons moving in the opposite direction to the current density $-\mathbf{j}$. In a plasma there are both positive and negative free charged ions, moving in opposite directions with different drift velocities, both of which contribute to the current density.

At room temperature, the average thermal velocity of an electron in a conductor is the order of $10^5 \mu = \sigma$. In contrast to this, the drift speed of electrons in a good conductor is very much smaller. Consider a 0.1 amp current in a 1mm diameter copper wire. The current density $j = \frac{0.1}{\pi(0.5)^2} = 127,323 A/m^2$. The number density of electrons in copper is about $8 \times 10^{28}$ electrons/m$^3$ and $q = -1.602 \times 10^{-19} C$. That is, $nq \approx 10^{10} C/m^3$. These give a drift speed $\mathbf{v} = \frac{\mathbf{j}}{nq} = 10^{-5} m/s = 3.6 cm/hour$, that is, $10^{-10}$ of the thermal velocity of individual electrons. This remarkably small drift velocity is a consequence of the very large charge density in a metal which implies small drift velocities for typical electric currents.

**CHARGE CONSERVATION**

One of the basic laws of nature, discussed in chapter 1, is the conservation of charge. For a current flowing out of a closed surface, the conservation of charge implies that the net current out must equal the rate of loss of charge from the enclosed volume. For any arbitrary closed surface this can be expressed as:

$$\oint_{\text{Closed surface}} \mathbf{j} \cdot d\mathbf{S} + \frac{d}{dt} \int_{\text{Enclosed volume}} \rho d\tau = 0$$

This statement of charge conservation, that relates charge and current densities, is extremely important. As you will see later it is a crucial and integral part of the Maxwell equations.

**ELECTRIC CONDUCTIVITY**

Experimentally it is found in most materials that the current density is approximately proportional to the applied electric field. This can be written as:

$$\mathbf{j} = \sigma \mathbf{E}$$

where the constant of proportionality $\sigma$ is called the conductivity. The experimental fact that the conductivity is a fairly good constant independent of $E$ for many materials, is called Ohms Law. It is unfortunate that this is called a law, since it is not a fundamental physical law of nature, It is just an approximate linear behavior of many materials. For non-ohmic materials, the conductivity does depend on $E$. The proportionality of current density and electric field can be expressed in terms of the resistivity $\rho$:

$$\mathbf{j} = \frac{1}{\rho} \mathbf{E}$$

where the conductivity $\sigma$ and resistivity $\rho$ are related by:

$$\sigma = \frac{1}{\rho}$$

Again, Ohm’s Law corresponds to assuming that $\rho$ is independent of the electric field.

Consider a cylindrical wire of cross sectional area $A$, length $d$, of uniform conductivity $\sigma$, in a uniform electric field $E$. Using Ohm’s Law the current in the wire is:

$$I = \int_{A} \mathbf{j} \cdot d\mathbf{S} = \sigma EA$$

The potential difference across the wire is:

$$\Delta V = \int \mathbf{E} \cdot d\mathbf{l} = Ed$$
Eliminating E gives:

$$\Delta V = \left(\frac{d}{A\sigma}\right)I$$

Define resistance R as:

$$R \equiv \frac{d}{A\sigma} = \frac{\rho d}{A}$$

Ohm’s law states that the conductivity, or equivalently resistivity, are independent of E for many materials. That is the resistance R is a constant independent of voltage difference. Usually, Ohm’s law is written as:

$$V = IR$$

To conform with the usual notation, the potential difference across the resistor, $\Delta V$, will be written as V which is assumed to imply voltage difference, not voltage relative to some reference potential.

The unit of resistance is the Ohm = Volts/amperes. Thus the unit of resistivity $\rho$ is ohm-m. It is convention to abbreviate the Ohm using the symbol $\Omega$.

**MICROSCOPIC PICTURE OF CONDUCTION**

Since $\vec{j} = nq\vec{v}$, then Ohm’s law, $\vec{j} = \sigma\vec{E}$, implies that the average drift velocity $\vec{v} = \frac{q\vec{E}}{m}$. This is surprising in that, Newton’s Law, states that $\vec{F} = q\vec{E} = m\vec{a}$, that is, the acceleration $\vec{a} = \frac{q\vec{E}}{m}$ in conflict with what Ohm’s law implies.

The explanation for this behaviour can be found in the picture of a gas of free electrons moving in a crystal lattice. After a collision, the electron starting with initial velocity $\vec{v}_0$ will accelerate in the E field resulting in an average velocity $\langle \vec{v} \rangle$ at average collision time $< t >$ of

$$\langle \vec{v} \rangle = \langle \vec{v}_0 \rangle + \frac{q\vec{E}}{m} < t >$$

The average $< \vec{v}_0 \rangle$ is zero since the electron gas is moving in all directions with equal probability. Thus the average drift velocity at the average collision time $< t >$ is

$$\langle \vec{v} \rangle = \frac{q < t > \vec{E}}{m}$$

Thus the current density is given by:

$$\vec{j} = nq\vec{v} = n\frac{q^2 < t > \vec{E}}{m}$$

That is the conductivity is:

$$\sigma = \frac{1}{\rho} = \frac{nq^2 < t >}{m}$$

This relation implies Ohm’s law in that the conductivity is independent of electric field. This proof assumed that, in collisions with the crystal lattice, the electron loses knowledge of the drift velocity and rebounds isotropically. This is understandable in view of the factor of $10^{10}$ difference between the average thermal velocity and the drift velocity of electrons. The uniform drift velocity of a ball in a pin-ball machine is an example of this behaviour. Another example is the constant speed of falling rain drops which reach a terminal velocity due to the viscous drag of the air. Without this air drag, the droplets could strike you at around 300 mph which would not be a pleasant experience.

The kinetic energy from acceleration of electrons in the electric field is dissipated as heat in the frequent collisions. This explains why the temperature of conductors in increased when an electric current flows. The resistance of conductors increases with increase in temperature because the larger amplitude of thermal motion of the atoms reduces the mean free path between collisions and the time $< t >$ to move the mean free path between collisions of the faster-moving electrons is lower. This temperature dependence is illustrated in Figure 5. The above free-electron model of conduction is inadequate to fully explain conduction.
One needs to use quantum physics for a full explanation.

Figure 6 lists the resistivities, conductivities and temperature dependence for many commonly used materials. Note that the resistivities range in value over twenty orders of magnitude.

At very low temperatures, certain materials lose all of their resistance abruptly at a critical temperature. When the material has a zero resistance it is called a superconductor. Superconductors have remarkable properties with interesting applications. Most materials are superconductors only at very low temperatures; typically $< 20^0K$. During the past couple of decades, a new class of superconductors has been discovered that become superconducting at temperatures around $200^0 K$, that is at liquid nitrogen temperatures. This new class of superconductors is potentially of great importance to science and engineering. Electric currents in a superconductor can persist for up to $10^9$ years. The explanation of superconductivity and possible applications requires knowledge of quantum physics and thus cannot be discussed usefully at this stage.

### ELECTRIC POWER

Resistance in a conductor is analogous to drag in that the electric energy derived from acceleration of charges in the electric field is converted to thermal energy. The potential energy lost can be calculated since we know that a charge $\Delta Q$ moving through a potential difference $V$ loses an energy:

$$\Delta U = \Delta QV$$

This energy is converted to heat. It follows that the power dissipated as heat $P$ is:

$$P = \frac{dU}{dt} = V \frac{dQ}{dt}$$

However, the electric current $I = \frac{dQ}{dt}$. Thus we have that the electric power dissipated is:

$$P = VI$$

This is generally true whether the material is ohmic or non-ohmic.

The units of power are $watt = volts \times amperes$. Note that the watt is the rate of energy dissipation, that is, $watt = \frac{joules}{second}$ . You are familiar with the term watts. The watt is the power generated when a force of 1 N displaces an object with a speed of 1m/s. Another common unit of power is the horsepower = 746W. At full power my 300 hp car engine produces $300 \times 746 = 223.8kW$ of power.

The total energy dissipated is just the product of power and time interval. The normal unit of work is the joule. However, another energy unit that is often used is the kW-Hour. The kilowatt-hour= $3.6 \times 10^6 J$. For example, a 100W bulb will burn $100 \times 3600 \times 24 = 8,640,000 J$ of energy per day, or 2.4kW−hr per day. If the power costs 8 cents/kW-hr, then the 100W bulb costs 0.8 cents per hour, or 19.2 cents per day, to operate. A typical power station can produce $300MW$ of power.

Note that for ohmic materials one can use $V = IR$ in the power relation to get that the power dissipated is:

$$P = I^2R = \frac{V^2}{R}$$

Thus if a 100W light bulb dissipates 100W of energy, and if the potential difference is 100V, then it must draw 1 amper. Note that in Europe, power in the home is distributed at 240V rather than the 110V used in the USA. For a given power demand by some appliance, the current needed is $I = \frac{P}{V}$. The power lost due to the resistance $R$ in the wires carrying the current to the appliance is much lower for the higher voltage distribution system because of the reduced current. It is for this reason that long-distance power distribution utilizes 500kV. For safety this high voltage is transformed to a lower voltage before local distribution to users.

### ELECTROMOTIVE FORCE

For electrostatics we know that current will flow in a conductor until a charge distribution builds up to cancel the applied electric field in the conductor. However, if the conductor is connected to a power source, such as a battery, then the voltage difference across
the conductor, and the resultant electric current will be maintained. The power source supplies a certain voltage difference, $\xi$, across the output terminals even when a current is drawn. That is, for a current $I$, the power supply provides $\xi I$ watts of power. In the power source, energy is derived from; chemical energy for a battery, mechanical energy in a Van de Graaff, kinetic energy of falling water in a hydroelectric power station, and sunlight in a solar cell. The voltage provided by the power source is called the electromotive force, usually abbreviated as emf. The source of the emf can be thought of as a charge pump that pumps charge from a region of low electric potential to one at a higher potential. That is, it pumps charge up through a potential difference of $\xi$ volts.

An ideal power source will maintain the emf across the terminals independent of the current flow. In practice, there is an internal resistance to the power source causing the effective emf to vary with current. Consider the simple circuit shown in figure 7. The power source, with emf $\xi$, and internal resistance $r$, is connected to a resistor $R$.

The potential at point $a$ is related to that at point $b$ by:

$$V_a = V_b + \xi - Ir$$

The voltage difference $V$ across the terminals is

$$V = V_a - V_b = \xi - Ir$$

This voltage difference must equal the voltage drop across the external resistor:

$$V = IR = \xi - Ir$$

Thus we have:

$$I = \frac{\xi}{r + R}$$

The typical internal resistance of a 12V car battery is $<0.01 \Omega$, whereas a flashlight battery has an internal resistance of about $0.1\Omega$. Thus the car battery can provide the order of 100 amps needed to start a car engine. The indication of a bad battery is a high internal resistance. A bad car battery may read 12V when a low current voltage meter is used, whereas the voltage across the battery will drop significantly when a large current is drawn by an external circuit.

**KIRCHHOFF’S RULES**

These apply to systems having steady currents and charges.

**Loop Rule:** The algebraic sum of all potential differences around a closed circuit is zero. This is Maxwell’s statement that for steady currents, around a closed circuit:

$$\int E \cdot dl = 0$$

That is, going clockwise around the circuit given in figure 7, the voltage drop across the resistors balances the emf:

$$\xi - Ir - IR = 0$$

Obviously the electric field $E$ points downward between the terminals of the source of emf, both via the power source and via the resistors.

**Node Rule:** The algebraic sum of all currents leaving a node is zero for a steady state system. This is a statement of the conservation of charge, that is, the net charge flowing into a junction must equal the net charge flowing out, if there is no source or sink of current at the junction.

Kirchhoff’s Rules are applicable to both direct current, DC, and alternate current, AC, circuits. However, handling of AC circuits requires knowledge of both the amplitude, frequency and phase of the currents and voltages which becomes mathematically complicated. Thus at this stage it is better to focus on DC circuits.

Applying Kirchhoff’s Rules is straightforward. The first step is to arbitrarily assign a label and direction to the current in each leg of the circuit and draw this on the circuit diagram. It does not matter which direction you assume that each current flows, the derived current will be negative if you chose the wrong direction.

**DC CIRCUITS**

As a first example consider the following series and parallel resistor circuits shown in figure 8.

**Resistors in series:**

Applying the Node rule is simple, obviously there is only one current $I$. Applying the Loop Rule to the three series resistors shown in figure 8 gives:

$$IR_1 + IR_2 + IR_3 - \xi = 0$$

That is:
\[ \xi = I(R_1 + R_2 + R_3) = IR_{total} \]

That is:

\[ R_{Total} = \sum_i R_i \]

**Resistors in parallel:**

Applying the Node Rule to the parallel system of three resistors, shown in figure 8, implies that the sum of the currents through the resistors equals the total current from the power source. That is:

\[ I_{Total} = I_1 + I_2 + I_3 \]

Also the voltage drop across each resistor must equal \( \xi \) from the Loop Rule, that is:

\[ \xi = I_1R_1 = I_2R_2 = I_3R_3 \]

Thus:

\[ I_{Total} = \xi \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) = \frac{\xi}{R_{Total}} \]

That is:

\[ \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{R_{Total}} \]

**Jump start car:**

A common problem in colder climates is having to jump start a car that has a dead battery as illustrated in figure 9. Clearly there is a correct way and a dangerous wrong way to connect the batteries. Assume that the good battery has an emf of \( \xi_1 = 12 \text{V} \). Assume the bad battery has an emf of \( \xi_2 = 11 \text{V} \). Let the internal resistance of both batteries is \( r_1 = r_2 = 0.02 \Omega \) while the resistance of the jumper cables is \( R = 0.01 \Omega \).

The correct way to connect that batteries is positive to positive and negative to negative as shown in the upper circuit of Fig 9. In this case the charging current is given by

\[ I = \frac{\xi_1 - \xi_2}{R + r_1 + r_2} = \frac{12 - 11}{0.05} = 20 \text{A} \]

The wrong way to connect that batteries is positive to negative and negative to positive as shown in the lower circuit of Fig 9. In this case the charging current is given by

\[ I = \frac{\xi_1 + \xi_2}{R + r_1 + r_2} = \frac{12 + 11}{0.05} = 460 \text{A} \]

Connected the wrong way the batteries may explode in a shower of sulphuric acid due to the large current draw.

**Two battery circuit**

The circuit shown in figure 10 is interesting to consider since there are two batteries involved as well as three resistors. Let us first derive the currents in the resistors. The first step is to choose symbols and directions for the currents in each leg.
Apply the Kirchhoff’s Node rule gives

\[ I = I_1 + I_2 \]

Apply the Loop rule to the loop abedef

\[ 12 - 2I_2 - 5 - 3(I_1 + I_2) = 0 \]

Apply the Loop rule to the loop abef

\[ 12 - 4I_1 - 3(I_1 + I_2) = 0 \]

Solving these three equations gives that

\[ I_1 = 1.5A \]
\[ I_2 = 0.5A \]

Now it is interesting to calculate the power provided by the batteries and dissipated in the resistors. The power dissipated in a resistor is given by

\[ P_R = I^2R \]

Thus the power dissipated is:

\[ P_4 = I_4^2 = 9.0\text{watts} \]
\[ P_2 = I_2^2 = 0.5\text{watts} \]
\[ P_3 = (I_1 + I_2)^2 = 12.0\text{watts} \]

Thus the total power dissipated in the resistors is

\[ P_4 + P_2 + P_3 = 21.5\text{watts} \]

The power provided by the 12V battery is given by

\[ P_{12} = (I_1 + I_2)12 = 24\text{watts} \]

The power provided by the 5V battery is

\[ P_5 = -I_212 = -2.5\text{watts} \]

That is the 5V battery is being charge by absorbing 2.5\text{watts} power.

Wheatstone Bridge:
A more complicated system used extensively in electrical instruments is the Wheatstone Bridge shown in figure 6. Using Kirchhoff’s Rules one gets six equations:

Node a:  \[ I_1 - I_2 - I_5 = 0 \]
Node b:  \[ I_2 - I_3 - I_4 = 0 \]
Node c:  \[ I_4 + I_5 - I_6 = 0 \]
Loop 1:  \[ \xi - I_1R_1 - I_2R_2 - I_5R_3 = 0 \]
Loop 2:  \[ -I_3R_5 + I_4R_4 + I_2R_2 = 0 \]
Loop 3:  \[ -I_6R_6 + I_3R_3 - I_4R_4 = 0 \]

These six equations can be solved for the six unknowns. The interesting case is when it is used as a bridge, where \( R_4 \) is a sensitive meter and the ratio of \( R_5 \) and \( R_6 \) are adjusted until the meter reads \( I_4 \) = 0. For this null technique \( I_2 = I_3 \) and \( I_5 = I_6 \). Therefore:

\[ \frac{R_2}{R_3} = \frac{R_5}{R_6} \]

If the resistance of three of the resistors is known, then the fourth can be deduced. Standard bridges for measuring resistance use the Wheatstone bridge. The current in meter \( R_4 \) is very sensitive to small differences in resistances in the bridge, a feature used extensively in instrumentation.

**SUMMARY**

The concept of electric current \( I \) and current density \( \mathbf{j} \) have been introduced. Ohm’s law, an approximate linear behavior of many materials is

\[ \mathbf{j} = \sigma \mathbf{E} \]

where the conductivity \( \sigma \) and resistivity \( \rho \), are related by \( \sigma = \frac{1}{\rho} \).

For a resistor \( R \), Ohm’s law can be expressed as

\[ V = RI \]

Electric power dissipated in a circuit carrying current \( I \) and voltage drop \( V \) is \( P = VI \).

Kirchhoff’s Rules are:

Loop Rule: The algebraic sum of all potential differences around a closed circuit is zero.

Node Rule: The algebraic sum of all currents leaving a node is zero for a steady state system.

Kirchhoff’s rules were used to solve the response of DC circuits.

**Reading assignment:** Giancoli Chapters 25, 26.1-26.4.