19. Lecture, 4 November 1999

19.1 Noise in detection

Since astronomical objects are so faint, the discussion of astronomical detection begins with considerations of *noise*: how quantum fluctuations determine the smallest signal that can be detected. Random fluctuations are imposed on the output of circuits, electromagnetic radiation, and the detection mechanism for two main reasons:

1. Quantization (usually the finite charge of the electron), and

2. The uncertainty principle.

We will first consider two aspects of noise for which the finite charge of the electron is responsible: *shot noise* (also called *photon noise*, when applied to photodetectors) and *Johnson noise*. These concepts will be useful in all of our discussions of the fundamental limitations to detection.

19.2 Shot noise

Suppose a simple circuit, shown in Figure 19.1, has a *very small* but steady DC current I running through it, and that we can measure this current by counting the charge carriers (electrons) as they pass some point A. Because the electrons can collide with each other and with the charged metal ions in the wire, they pass point A at completely random intervals. They also have a finite charge, $q = -1.6022 \times 10^{-19}$ coulombs or -4.803×10^{-10} esu, so although after counting the electrons for *many* intervals of length Δt we would get an average number $\overline{N}$ of

$$\overline{N} = \frac{I\Delta t}{q} \quad , \tag{19.1}$$

during a period Δt, the number for any given Δt would be different; sometimes a few more, sometimes a few less. Thus we would rather speak about chance: what is the *probability* that exactly N (no bar) show up during a time interval Δt?

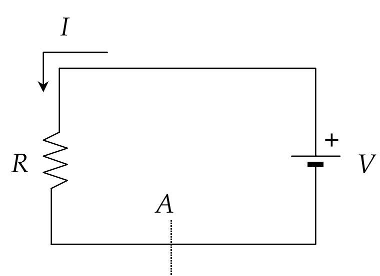

Figure 19.1: current with shot noise.

To answer this question, we first divide the *exposure time* Δt up into a very large number n of equal segments of time. The average number arriving in a segment is $\overline{N}/n$, and if n is very large, $\overline{N}/n << 1$. Let's use that limit. Since we can't have less than one electron arrive, what this means is that the *probability* that an electron arrives during one given segment is $\overline{N}/n$, where the probability is simply the number of times something is likely to occur per time it's tried; if something is sure to happen, its probability is 1. We also see from this limit that the probability of an electron *not* arriving during the given segment is $1-\overline{N}/n$, since there is a total probability of 1 that an electron will either show up or not show up, and that the probability of two electrons showing up during the same segment is $\left(\overline{N}/n\right)^2$ – so much rarer than single-electron arrivals that we will henceforth neglect multiple-electron events. The probability of the arrival of electrons within N segments is therefore $\left(\overline{N}/n\right)^N$, and the probability of no electrons arriving within the remaining n-N segments is $\left(1-\overline{N}/n\right)^{n-N}$.

There are many ways in which N electrons can arrive within n time segments: there are $n!$ different rearrangements of the segments [1], N of which contain electron arrivals. On the other hand electrons are indistinguishable particles, so any of these arrangements in which electrons arrived in the same set of segments but in a different order are actually indistinguishable from one another; since there are $N!$ arrangements of N electrons, the number of different arrangements of the n segments must be reduced by a factor of $N!$. Similarly, the n-N segments in which no electrons arrive are indistinguishable, and the number of different arrangements must be reduced further by a factor of (n-N). Thus the number of *different* ways in which N electron arrivals can be spread over n time segments is $n!/N!(n-N)!$, the probability that exactly N electrons arrive during the interval Δt is

$$p_n(N) = \frac{n!}{N!(n-N)!}\left(\frac{\overline{N}}{n}\right)^N\left(1-\frac{\overline{N}}{n}\right)^{n-N} \quad . \tag{19.2}$$

This probability function is called the *binomial distribution*; some experience with it will be gained in the course of Homework Problem 19.1. If it is correct, then the probabilities for each N from zero to n should add up to 1, so we should check:

$$\begin{aligned}\sum_{N=0}^{n} p_n(N) &= \sum_{N=0}^{n}\frac{n!}{N!(n-N)!}\left(\frac{\overline{N}}{n}\right)^N\left(1-\frac{\overline{N}}{n}\right)^{n-N} \\ &= \left(1-\frac{\overline{N}}{n}\right)^n\sum_{N=0}^{n}\frac{n!}{N!(n-N)!}\left(\frac{\overline{N}}{n}\right)^N\left(1-\frac{\overline{N}}{n}\right)^{-N} \quad .\end{aligned} \tag{19.3}$$

Recall the binomial theorem, Equation 7.9:

$$(1+x)^s = \sum_{n=0}^{\infty}\frac{s!}{n!(s-n)!}x^n \qquad \left(|x|<1\right), \tag{19.4}$$

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[1] Consider the number of choices available in the arrangement of n distinguishable objects of any sort. There are n possibilities for the first choice, since any can be chosen; n-1 for the second choice, which must be made among the remaining ones; n-2 for the third choice; and so on, with only one possibility for the nth choice. The number of different arrangements of the objects is the product of all of the numbers of possibilities, or $n(n-1)(n-2)\ldots(2)(1) = n!$.

and let $x = \left(\frac{\overline{N}}{n}\right)\left(1-\frac{\overline{N}}{n}\right)^{-1}$, so that the sum in Equation 19.3 is given by 19.4:

$$\sum_{N=0}^{n} p_n(N) = \left(1-\frac{\overline{N}}{n}\right)^n \left[1+\left(\frac{\overline{N}}{n}\right)\left(1-\frac{\overline{N}}{n}\right)^{-1}\right]^n = \left[\left(1-\frac{\overline{N}}{n}\right)+\left(\frac{\overline{N}}{n}\right)\right]^n = 1 \quad . \tag{19.5}$$

(It checks out.)

Homework problem 19.1. *Noise in a random walk.* A friend of yours has just left your party, in a world-record state of drunkenness. She can't drive home, because you've wisely hidden her car keys, but her walking suffers from the following restriction: to hold herself up she leans her back against the wall of the building, so all she can do is stagger along in one dimension. Every lurching step she takes is one meter long. The directions of her steps are of course completely random; suppose the probability of any given step being in her right-hand direction is q. You wish to estimate where she's likely to be as a function of how many steps she's taken.

a. Argue that the probability for her to take N steps to the right out of n total steps is governed by the binomial distribution, equation 19.5.

b. Derive an expression for her average distance from your door after n total steps.

c. Derive an expression for the rms deviation of her distance from your door, after n total steps.

d. Suppose she is equally likely to step to the left or the right. How far from your door is she likely to be after 120 steps? (Report this in the form $x \pm \Delta x$.) What is the probability that she'll be falling through your neighbor's open door, 40 meters to her right of yours, after 120 steps? *Hint:* remember Stirling's approximation for the factorial of a large number n:

$$\ln(n!) \approx n \ln n - n + \frac{1}{2}\ln(2\pi n) \quad .$$

e. Suppose now that the weight of her backpack, slung over her right shoulder, leans her over enough that she is twice as likely to take a step to the right as to the left. Now how far from your door is she likely to be after 120 steps? (Report it in the form $x \pm \Delta x$.) What is the probability now that she'll fall through your neighbor's door on her 120$^{\text{th}}$ step (if she hasn't already)?

Let us apply the binomial distribution to the case in which we want the time segments to be infinitesimal (i.e. $n \to \infty$). In Equation 19.2 we can cancel the $(n-N)!$ in the denominator with the numerator, leaving N factors in the numerator that can be arranged suggestively with the factor $1/n^N$:

$$\begin{aligned} p(N) &= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-N+1)}{N!}\left(\frac{\overline{N}}{n}\right)^N \left(1-\frac{\overline{N}}{n}\right)^{n-N} \\ &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{N-1}{n}\right)\frac{\overline{N}^N}{N!}\left(1-\frac{\overline{N}}{n}\right)^{n-N} \quad . \end{aligned} \tag{19.6}$$

All of the leading terms here become 1 in the limit $n \to \infty$ $(1/n \to 0)$, so we are left with

$$p(N) = \frac{\overline{N}^N}{N!} \lim_{n\to\infty} \left(1 - \frac{\overline{N}}{n}\right)^{n-N} \quad . \tag{19.7}$$

To proceed we also need to recall the power-series expansion for the exponential,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(all } x\text{)} . \tag{19.8}$$

Now operate on the last term in Equation 19.4:

$$\begin{aligned} \lim_{n\to\infty} \left(1 - \frac{\overline{N}}{n}\right)^{n-N} &= \lim_{n\to\infty} \left(1 - \frac{\overline{N}}{n}\right)^{n} = \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{n!}{i!(n-i)} \frac{(-\overline{N})^i}{n^i} \\ &= \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-i+1)}{n^i} \frac{\left(-\overline{N}\right)^i}{i!} \quad . \end{aligned} \tag{19.9}$$

As before, there are i terms in the numerator of the first term in the sum, and i factors of n in the denominator:

$$\begin{aligned} \lim_{n\to\infty} \left(1 - \frac{\overline{N}}{n}\right)^{n-N} &= \lim_{n\to\infty} \sum_{i=0}^{\infty} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{1-1}{n}\right) \frac{(-\overline{N})^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{(-\overline{N})^i}{i!} = e^{-\overline{N}} \quad . \end{aligned} \tag{19.10}$$

Thus

$$p(N) = \frac{\overline{N}^N}{N!} e^{-\overline{N}} \tag{19.11}$$

is the probability that N electrons will arrive during Δt, given that the average number arriving during intervals of equal length is $\overline{N}$. This important result is called the *Poisson probability distribution*. . The Poisson distribution is plotted as a function of N in Figure 19.2. Note that the "curves" peak near, but not quite on, the average values; peak and average get closer for larger $\overline{N}$).

We expect, *a priori*, that the number of electrons lies somewhere between zero and infinity, so the sum of all the $p(N)$ from 0 to ∞ should add up to 1, and this turns out to work:

$$\sum_{n=\infty}^{\infty} p(N) = e^{-\overline{N}} \sum_{n=\infty}^{\infty} \frac{\overline{N}^N}{N!} = e^{-\overline{N}} e^{\overline{N}} = 1 \quad . \tag{19.12}$$

We also expect that the sum of all the products $Np(N)$ would give $\overline{N}$, since the probability $p(N)$ is the *fraction* of trials that would give the result N, and this also works:

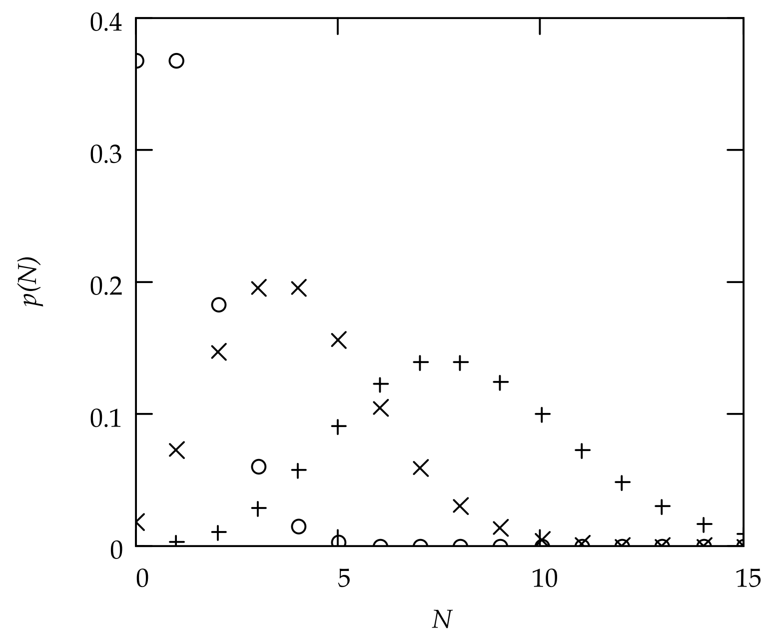

Figure 19.2: Poisson distribution for $\overline{N}=1$ (circles), 4 (xs) and 8 (+s).

$$\begin{aligned}\sum_{N=0}^{\infty} Np(N) &= e^{-\overline{N}} \sum_{N=0}^{\infty} N \frac{\overline{N}^N}{N!} = e^{-\overline{N}} \sum_{N=1}^{\infty} N \frac{\overline{N}^N}{N!} \\ &= \overline{N} e^{-\overline{N}} \sum_{N=1}^{\infty} \frac{\overline{N}^{N-1}}{(N-1)!} = \overline{N} e^{-\overline{N}} \sum_{M=0}^{\infty} \frac{\overline{N}^M}{M!} = \overline{N} e^{-\overline{N}} e^{\overline{N}} = \overline{N} \quad ,\end{aligned} \tag{19.13}$$

where we have made the substitution $M = N$-1 in the next to last step. In general, any function of N has an average value given by

$$\overline{f(N)} = \sum_{N=0}^{\infty} f(N)p(N) \quad . \tag{19.14}$$

All the statistics of the number of electrons arriving at point A, and any function that depends upon that number, are completely determined by the probability distribution $p(N)$. Let's do just one more function, $f(N) = N^2$:

$$\begin{aligned}\overline{N^2} &= \sum_{N=0}^{\infty} N^2 p(N) = \sum_{N=0}^{\infty} \left[N + N(N-1)\right] p(N) \\ &= \overline{N} + \sum_{N=0}^{\infty} N(N-1) \frac{\overline{N}^N}{N!} e^{-\overline{N}} = \overline{N} + e^{-\overline{N}} \sum_{N=2}^{\infty} N(N-1) \frac{\overline{N}^N}{N!} \\ &= \overline{N} + \overline{N}^2 e^{-\overline{N}} \sum_{N=2}^{\infty} \frac{\overline{N}^{N-2}}{(N-2)!} = \overline{N} + \overline{N}^2 e^{-\overline{N}} \sum_{M=0}^{\infty} \frac{\overline{N}^M}{M!} = \overline{N} + \overline{N}^2 \quad ,\end{aligned} \tag{19.15}$$

where we have substituted $M = N - 2$ in the penultimate step

Usually we don't need the probability distribution function in order to discuss how big the fluctuations might be - we keep track of it approximately by the mean value and the width of the curve. A standard measure of the width of the curve is the *standard,* or *root-mean square (rms), deviation from the mean*, which in turn is the square root of the *variance*, defined by

$$\begin{aligned}\overline{(\Delta N)^2} &\equiv \overline{\left(N-\overline{N}\right)^2} \\ &= \overline{N^2} - \overline{2N\overline{N}} + \overline{N}^2 \\ &= \overline{N^2} - \overline{N}^2 \quad .\end{aligned} \tag{19.16}$$

For the Poisson distribution, $\overline{N^2} = \overline{N} + \overline{N}^2$, as we showed above, so

$$\begin{aligned}\overline{(\Delta N)^2} &= \overline{N} \\ (\Delta N)_{\mathrm{rms}} &= \sqrt{\overline{N}} \quad .\end{aligned} \tag{19.17}$$

So, we could say that the number of electrons arriving at point *A* during Δt, averaged over a large number of exposures Δt long, is $\overline{N} \pm \sqrt{\overline{N}}$, which expresses the fact that the answer is in general different in different trials , but most of the time the number of arrivals is within $\pm\sqrt{\overline{N}}$ of $\overline{N}$.

Now back to our noisy current: the average current $I = \bar{I}$ is just $q\overline{N}/\Delta t$, but we can now characterize the *fluctuations* in the current about its mean value obtained in successive trials as a *noise current,* I_N:

$$\begin{aligned}I_N^2 &= \overline{(\Delta I)^2} = \frac{q^2}{\Delta t^2}\overline{\left(N-\overline{N}\right)^2} = \frac{q^2\overline{N}}{\Delta t^2} \\ &= \frac{qI}{\Delta t} \quad ,\end{aligned} \tag{19.18}$$

where we have used Equation 19.16 in the next to last step. The important things to notice about this result, first derived by Schottky in 1916, are that *the rms fluctuations of current are proportional to the square root of current*, and that *the rms fluctuations decrease proportionally as the square root of the exposure time increases.* The implication of the latter is obvious. Suppose there is a certain shot-noise-dominated current, for which it is desired to decrease the rms fluctuations by a factor of two, say, in order to look for a small signal current. One can do so by increasing the exposure time of the measurements of current by a factor of four.

19.3 Exposure time and bandwidth

In the literature, Equation 19.18 is expressed more often in terms of a frequency bandwidth associated with the exposure time Δt, than in terms of this time itself. To derive the bandwidth Δf which corresponds to Δt, we first note that the current measured as a function of time, by an observer watching point *A* in the circuit of Figure 19.1, can be written as a Fourier integral:

$$I(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(\omega)e^{i\omega t}d\omega \quad , \tag{19.19}$$

where $\omega = 2\pi f$ is the angular frequency. The Fourier components $a(\omega)$ can be determined by performing the inverse transform. Suppose that an exposure of length Δt has resulted in a value $\bar{I}$ for the average current. The Fourier components that would give a constant current $\bar{I}$ over the exposure time are

$$\begin{aligned} a(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} I(t)e^{-i\omega t}dt = \frac{\bar{I}}{\sqrt{2\pi}}\int_{-\Delta t/2}^{\Delta t/2} e^{-i\omega t}dt = \frac{\bar{I}}{\sqrt{2\pi}}\frac{e^{-i\omega\Delta t/2} - e^{i\omega\Delta t/2}}{-i\omega} \\ &= \frac{\bar{I}\Delta t}{\sqrt{2\pi}}\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2} \quad . \end{aligned} \tag{19.20}$$

We can use Rayleigh's theorem, Equation 12.17 (reduced to one dimension, that is) to write

$$\begin{aligned} \int_{-\infty}^{\infty} |I(t)|^2 dt &= \int_{-\infty}^{\infty} |a(\omega)|^2 d\omega \quad \text{, or} \\ \bar{I}^2\Delta t &= \frac{\bar{I}^2\Delta t^2}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2}\right)^2 d\omega \quad . \end{aligned} \tag{19.21}$$

The integral in this expression can be evaluated (giving a trivial result), but instead of doing so we note that it is evident here that the angular frequency bandwidth corresponding to the exposure time is given by

$$2\Delta\omega = \int_{-\infty}^{\infty}\left(\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2}\right)^2 d\omega \quad , \tag{19.22}$$

where the factor of two on the left makes up for the integration over negative frequencies as well as positive ones. This can be put into Equation 19.21, and the common factors cancelled therein, to yield

$$\begin{aligned} \Delta\omega\Delta t &= \pi \quad \text{, or} \\ \Delta f\Delta t &= \frac{1}{2} \quad . \end{aligned} \tag{19.23}$$

With this, the shot noise formula, Equation 19.18, becomes

$$I_N^2 = \overline{(\Delta I)^2} = 2qI\Delta f \quad , \tag{19.24}$$

or, in terms of the rms noise current,

$$I_N = \sqrt{2qI\Delta f} \quad . \tag{19.25}$$

The form of the results, Equations 19.18, 19.24-25, is due to Poissonian statistics, which in turn implies, as we assumed in the beginning, that the events (electron arrivals) are distinct and independent. This applies to a wide range of situations in which we will be interested, but we'll deal next with a system in which the arrivals are *correlated*, and Poissonian statistics do not apply.

