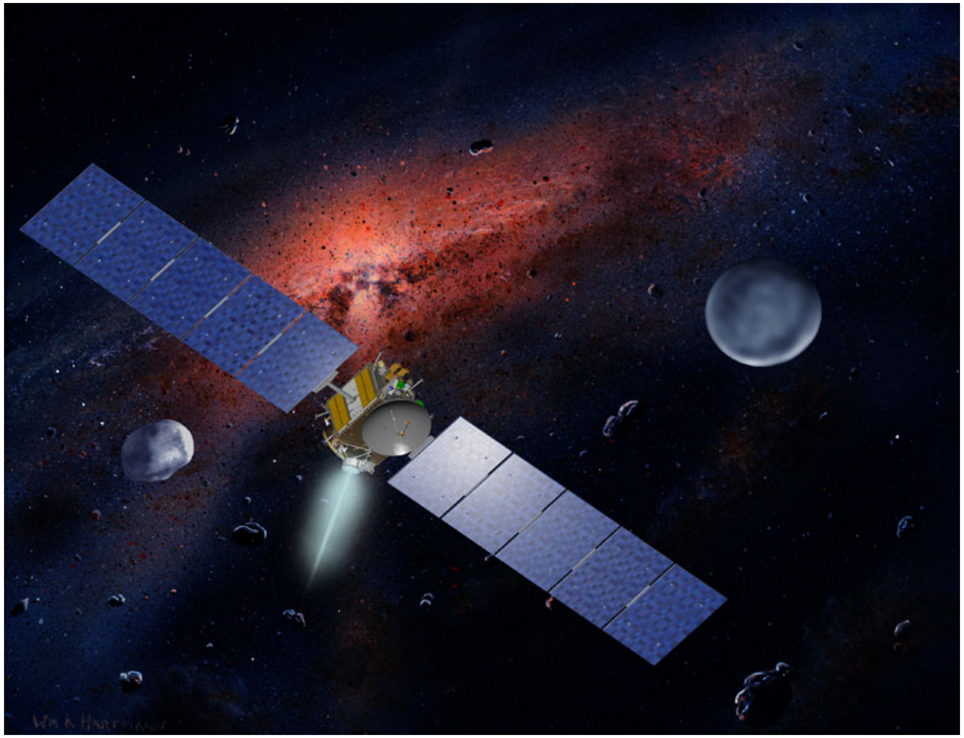
Today in Physics 122: Coulomb's-law examples and useful consequences

- ❑ Charged disk (from last time)
- ❑ Charged line segment; ***E*** everywhere
- ❑ Approximations and limiting cases
- ❑ Ion drive for space travel
- ❑ Ion drives *vs* rockets



NASA's *Dawn* spacecraft, which recently paid visits to asteroids 1 Ceres and 4 Vesta. (Artist's conception: William K. Hartmann and NASA)

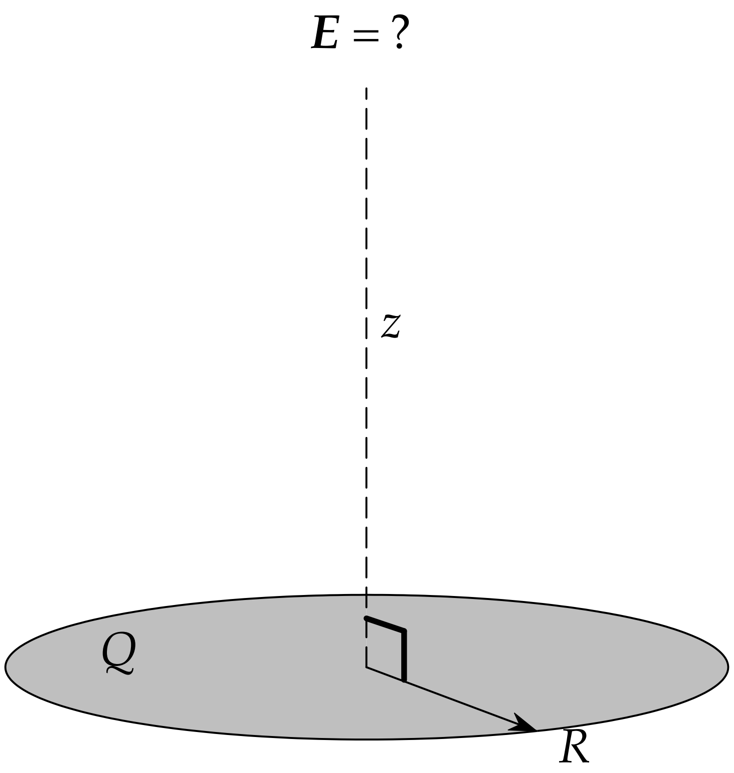
Example 2: charged disk

A circular disk *with radius R carries a charge Q which is uniformly distributed on the disk. Calculate the electric field a distance z above the center of the disk.*

- This time it's a two-dimensional charge density, though it's still the same density throughout the charge distribution:

$$\sigma = \frac{Q}{\pi R^2}$$

- The coordinate system from the previous example should be fine.

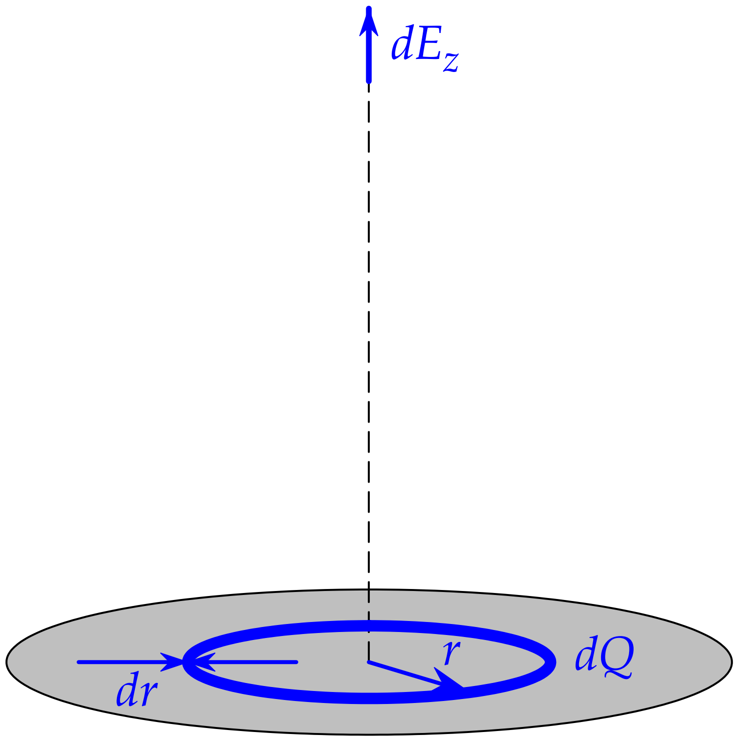


Example 2: charged disk (continued)

- And now for a useful trick: nothing obliges us to use an infinitesimal area element that's infinitesimal in all directions. Let's take as our area element an infinitesimally-thin annulus of radius r and width dr:

$$dA = 2\pi r dr$$

$$dQ = \sigma dA = 2\pi\sigma r dr$$



- We can regard this as a charged ring, for which we just calculated the field:

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{z\hat{\mathbf{z}}}{\left(z^2 + r^2\right)^{3/2}} dQ$$

Example 2: charged disk (continued)

- So we add the contributions of all the rings with radius varying between r = 0 and r = R:

$$\boldsymbol{E}(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{z\hat{\boldsymbol{z}}}{\left(z^2+r^2\right)^{3/2}}\,dQ = \frac{\sigma z\hat{\boldsymbol{z}}}{2\varepsilon_0}\int_0^R \frac{rdr}{\left(z^2+r^2\right)^{3/2}}$$

- You can look up this integral, as textbooks timidly recommend:

$$\int_0^R \frac{rdr}{\left(z^2+r^2\right)^{3/2}} = \left[-\frac{1}{\sqrt{z^2+r^2}}\right]_0^R = \frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}$$

(This is how you'd finish this problem on an exam, for full credit.)

Example 2: charged disk (continued)

- It isn't hard to do the integral oneself, as you learned in MTH 162, with a substitution of variables. One substitution that works is:

$$u = r^2 + z^2 \Rightarrow du = 2r\,dr$$

$$r = 0 \to R \Rightarrow u = z^2 \to z^2 + R^2$$

$$\int_0^R \frac{r\,dr}{\left(z^2 + r^2\right)^{3/2}} = \frac{1}{2}\int_{z^2}^{z^2+R^2} u^{-3/2}\,du = \frac{1}{2}\left[-2u^{-1/2}\right]_{z^2}^{z^2+R^2}$$

$$= \frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}}$$

Example 2: charged disk (continued)

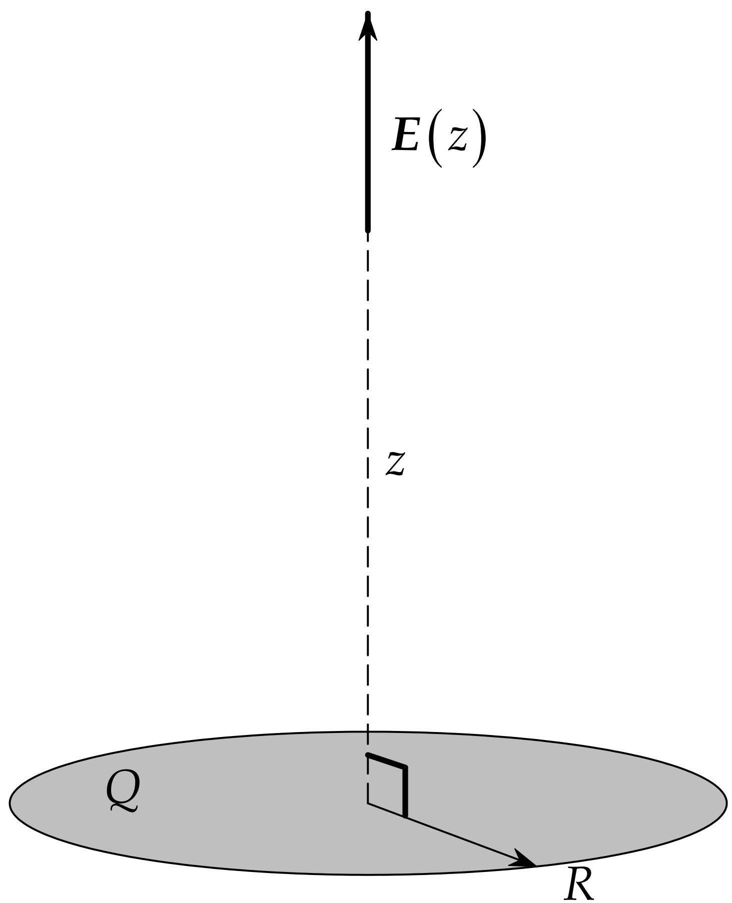
- Either way, we're ready to write down the final answer:

$$\boldsymbol{E}(z) = \frac{\sigma z \hat{\boldsymbol{z}}}{2\varepsilon_0} \int_0^R \frac{r dr}{\left(z^2 + r^2\right)^{3/2}}$$

$$= \frac{\sigma \hat{\boldsymbol{z}}}{2\varepsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)$$

or

$$= \frac{Q \hat{\boldsymbol{z}}}{2\pi\varepsilon_0 R^2} \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) \quad .$$



Example: field of a line charge

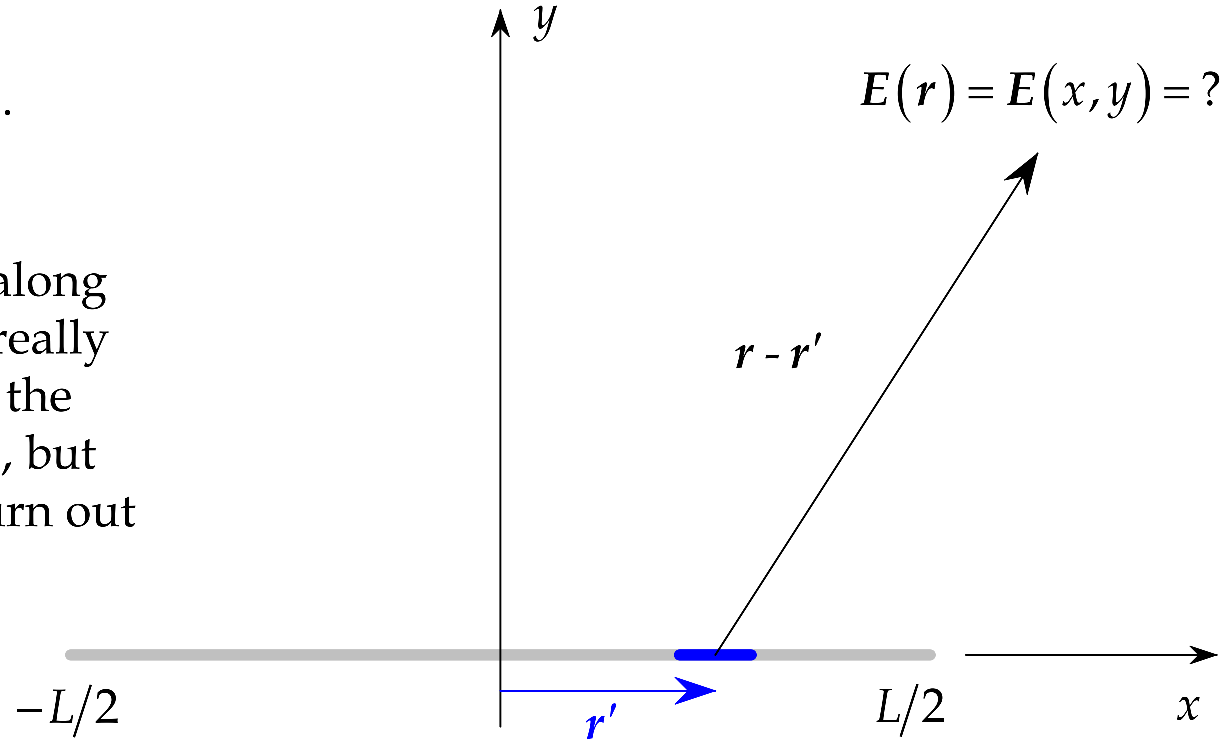
A straight wire of length L *carries a total charge* Q. *Calculate the electric field everywhere.*

Not quite like the workshop problem.

- Straight wires are one-dimensional:

$$\lambda = \frac{Q}{L} \quad .$$

- Better run one axis along the wire. It doesn't really matter where along the wire the origin goes, but the midpoint will turn out to be good.



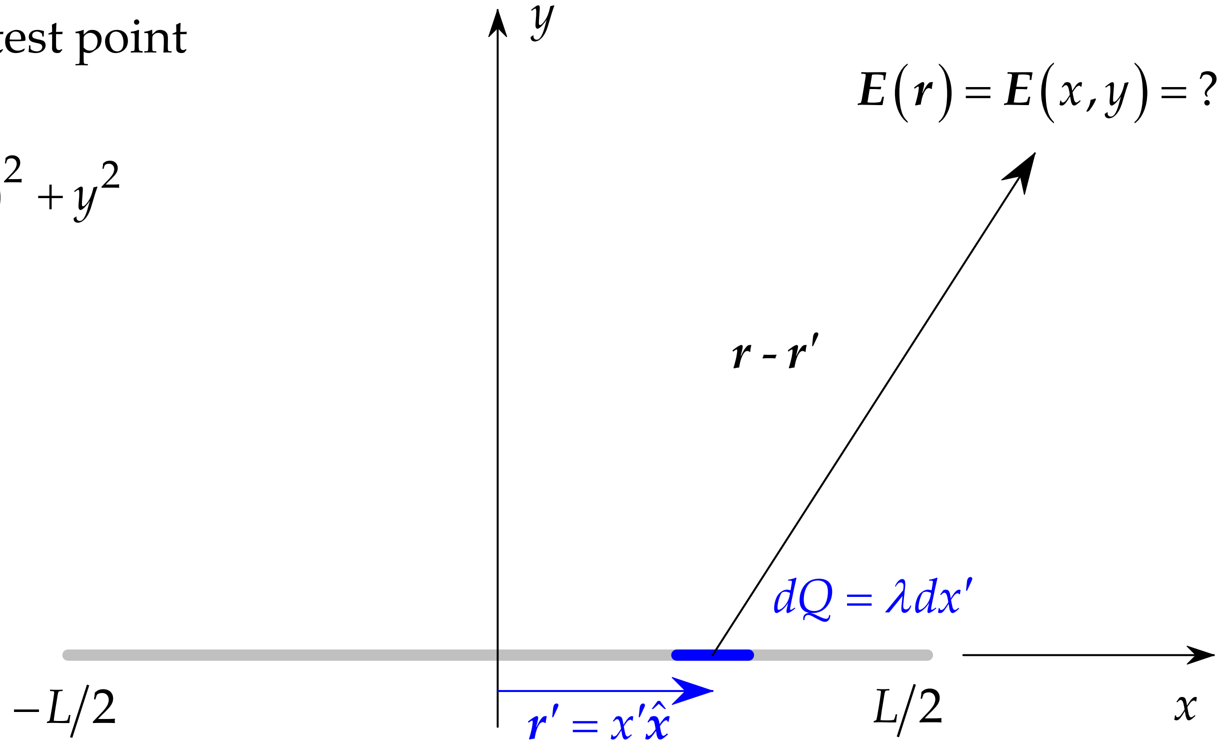
Field of a line charge (continued)

- The obvious choice for an infinitesimal element of length is dx', making $dQ = \lambda dx'$. Then the integral can run from

$$x' = -L/2 \text{ to } L/2.$$

- The distance to the test point from x' is such that

$$(\boldsymbol{r} - \boldsymbol{r}')^2 = (x - x')^2 + y^2$$



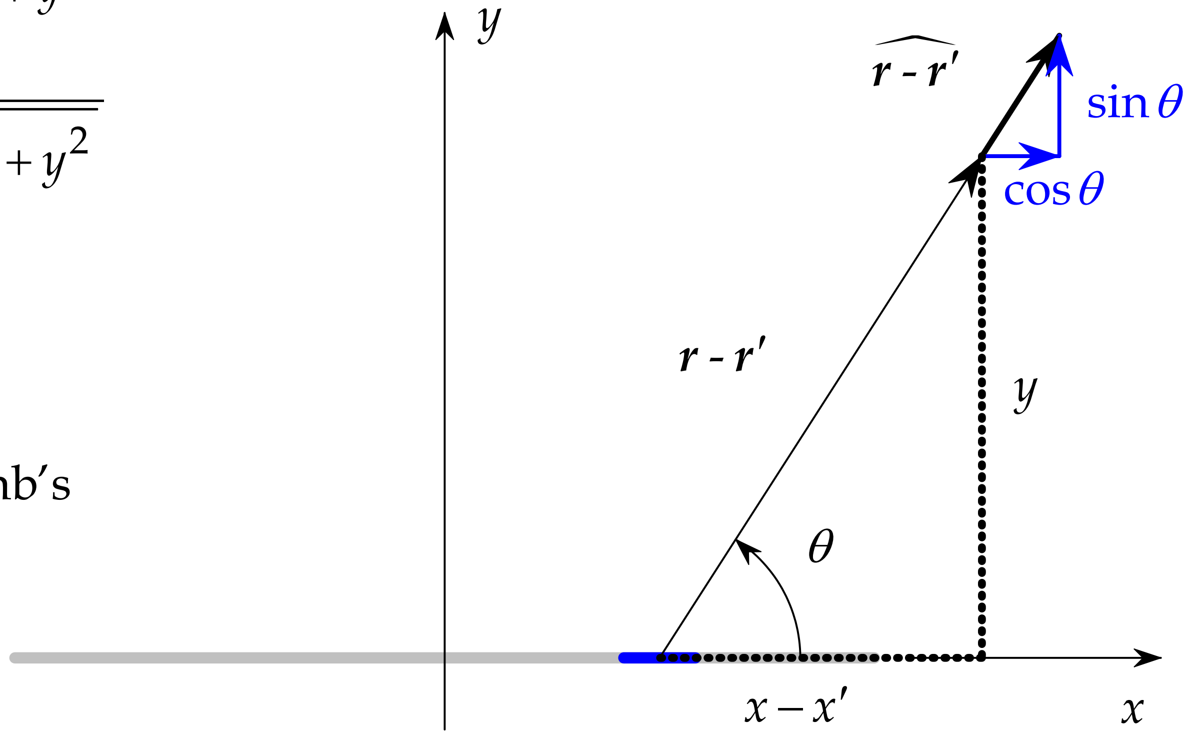
Field of a line charge (continued)

- From the triangle formed by the axes and $\boldsymbol{r}$ - $\boldsymbol{r}'$,

$$\sin\theta = \frac{y}{\sqrt{\left(x-x'\right)^2 + y^2}}$$

$$\cos\theta = \frac{x-x'}{\sqrt{\left(x-x'\right)^2 + y^2}}$$

- Now we write Coulomb's law for the field, one component at a time:



Field of a line charge (continued)

$$E_x = \int \frac{1}{4\pi\varepsilon_0} \frac{dQ}{\left(\boldsymbol{r}-\boldsymbol{r}'\right)^2}\cos\theta = \frac{1}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{\lambda dx'}{\left(x-x'\right)^2+y^2} \frac{x-x'}{\sqrt{\left(x-x'\right)^2+y^2}}$$

- With a minor variable substitution,

$$X = x - x' \quad \Rightarrow \quad dX = -dx'$$
$$x' = -L/2 \rightarrow L/2 \quad \Rightarrow \quad X = x + L/2 \rightarrow x - L/2$$

we see this is actually the integral we did with the charged disk:

$$E_x = -\frac{1}{4\pi\varepsilon_0}\lambda \int_{x+L/2}^{x-L/2} \frac{XdX}{\left(X^2+y^2\right)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{X^2+y^2}}\right]_{x+L/2}^{x-L/2}$$

Field of a line charge (continued)

or

$$E_x = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-L/2)^2+y^2}} - \frac{1}{\sqrt{(x+L/2)^2+y^2}}\right]$$

- Now the y component:

$$E_y = \int \frac{1}{4\pi\varepsilon_0}\frac{dQ}{(\boldsymbol{r}-\boldsymbol{r}')^2}\sin\theta = \frac{\lambda y}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{dx'}{\left((x-x')^2+y^2\right)^{3/2}}$$

In the spirit of the textbook, we can look up the integral in a table:

$$E_y = -\frac{\lambda}{4\pi\varepsilon_0}\frac{1}{y}\left[\frac{x-x'}{\sqrt{(x-x')^2+y^2}}\right]_{-L/2}^{L/2}$$

Field of a line charge (continued)

It's not that hard to integrate one's self: make a trigonometric substitution:

$$x - x' = y\tan\alpha \quad \Rightarrow$$

$$-dx' = y\left(1+\tan^2\alpha\right)d\alpha = \frac{y}{\cos^2\alpha}d\alpha$$

$$x' = -L/2 \to L/2 \quad \Rightarrow$$

$$\alpha = \arctan\left(\frac{x+L/2}{y}\right) \to \arctan\left(\frac{x-L/2}{y}\right) = \alpha_+ \to \alpha_-$$

The denominator under the integral becomes

$$\left(\left(x-x'\right)^2 + y^2\right)^{3/2} = \left(y^2\tan^2\alpha + y^2\right)^{3/2} = \frac{y^3}{\cos^3\alpha}$$

Field of a line charge (continued)

so $E_y = -\frac{1}{4\pi\varepsilon_0}\lambda y \int_{\alpha_+}^{\alpha_-} \frac{\cos^3\alpha}{y^3}\frac{y\,d\alpha}{\cos^2\alpha} = +\frac{\lambda}{4\pi\varepsilon_0}\frac{1}{y}\int_{\alpha_-}^{\alpha_+}\cos\alpha\,d\alpha = \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{y^2}\sin\alpha\Big|_{\alpha_-}^{\alpha_+}$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{y}\left[\sin\left(\arctan\left[\frac{x+L/2}{y}\right]\right) - \sin\left(\arctan\left[\frac{x-L/2}{y}\right]\right)\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{y}\left[\frac{x+L/2}{\sqrt{(x+L/2)^2+y^2}} - \frac{x-L/2}{\sqrt{(x-L/2)^2+y^2}}\right] \quad .$$

(Recall that if $\tan\theta = a/b$, then $\sin\theta = a\big/\sqrt{a^2+b^2}$ and $\cos\theta = b\big/\sqrt{a^2+b^2}$.)

Field of a line charge (continued)

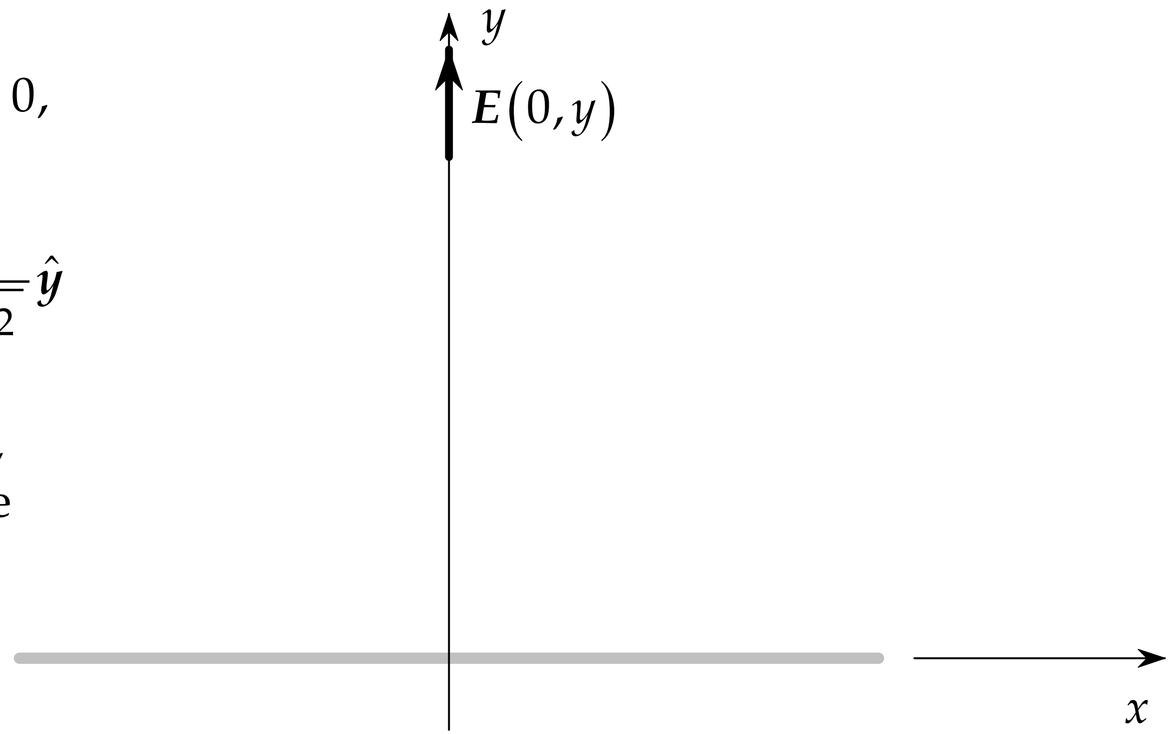
And we get

$$E_y = \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{y}\left[\frac{x+L/2}{\sqrt{\left(x+L/2\right)^2+y^2}} - \frac{x-L/2}{\sqrt{\left(x-L/2\right)^2+y^2}}\right]$$

- Note that if $x = 0$, $E_x = 0$, and thus

$$\boldsymbol{E} = \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{y}\frac{L}{\sqrt{\left(L/2\right)^2+y^2}}\hat{\boldsymbol{y}}$$

as found in workshop, and as you might have expected from the symmetry.



Other useful consequences of our recent Examples

1. Far away from a line charge

If the length L is very small compared to y, then the field in the plane through the midpoint looks even simpler:

$$\boldsymbol{E}(0,y)=\frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{y\sqrt{(L/2)^2+y^2}}\hat{\boldsymbol{y}}\cong\frac{1}{4\pi\varepsilon_0}\frac{Q}{y^2}\hat{\boldsymbol{y}}$$

$\lambda L \to Q$; $(L/2)^2$: neglect, compared to y^2.

just as it's supposed to from a place far enough away that the line looks like a point (Coulomb's law).

- Examining a limit in which you know what the answer is, is a good way of checking your answer.

Other useful consequences of our recent Examples (continued)

2. **Close to a line charge** (or anywhere, in the presence of an infinitely-long line charge)

   If on the other hand L is much larger than y,

$$\boldsymbol{E}(0,y) = \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{y\sqrt{(L/2)^2 + y^2}}\hat{\boldsymbol{y}} \cong \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{y}\hat{\boldsymbol{y}}$$

   neglect, compared to $(L/2)^2$

   E points perpendicularly away from a long line charge, and its magnitude drops off with increasing distance as $1/y$.

Other useful consequences of our recent Examples (continued)

3. Far from a charged disk.

For those familiar with the binomial approximation,
$(1+x)^n \cong 1+nx$ if $|x| \ll 1$:

$$\mathbf{E}(z) = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}\left(1 - \frac{1}{\sqrt{1+(R/z)^2}}\right)$$

$$\cong \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}\left(1 - \left[1 - \frac{1}{2}\left(\frac{R}{z}\right)^2\right]\right) = \frac{1}{4\pi\varepsilon_0}\frac{\pi R^2\sigma}{z^2}\hat{\mathbf{z}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}\hat{\mathbf{z}}$$, again recovering Coulomb's law.

Other useful consequences of our recent Examples (continued)

4. **Near a charged disk,** or in the presence of an infinite charged disk

If the radius R of the charged disk we considered last time is very large compared to z, then the field along its axis winds up not depending upon z at all:

neglect, compared to 1

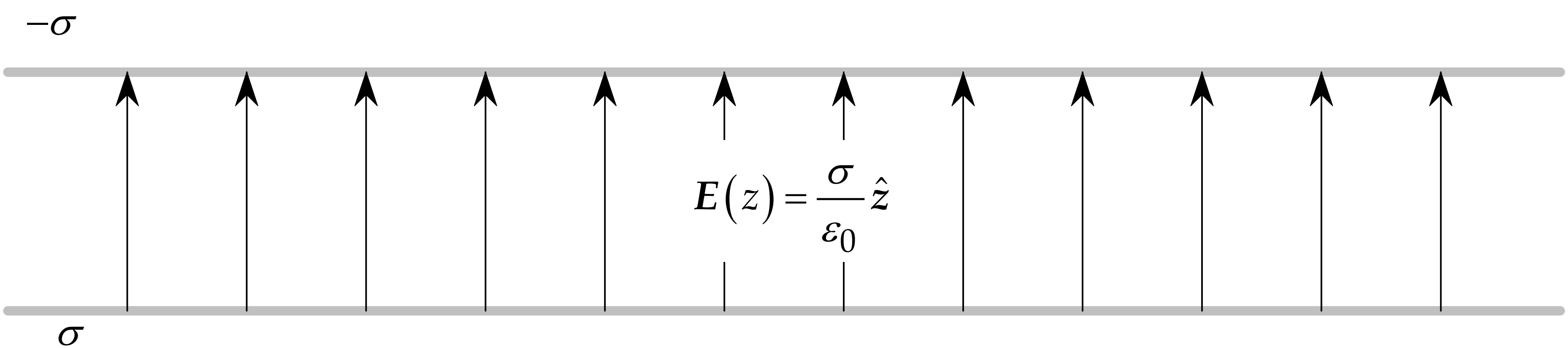
$$\boldsymbol{E}(z) = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{z}}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) \cong \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{z}}$$

neglect, compared to R^2

-- that is, a **uniform** electric field.

How to make a uniform electric field

This last result suggests a nice way to make a uniform electric field, should you ever need one in the lab: two uniformly-charged parallel planes, with distance small compared to length or width. Seen edge on:

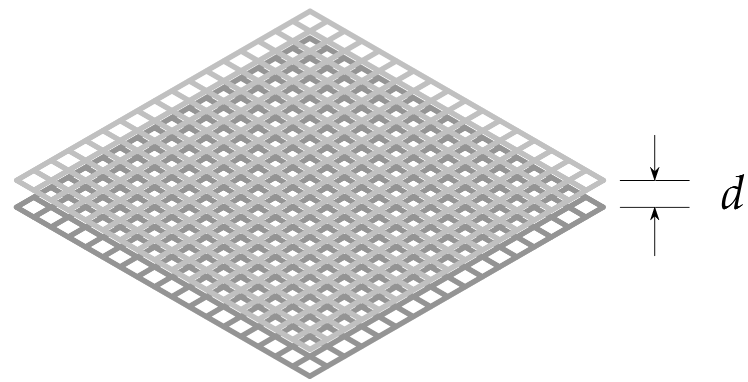


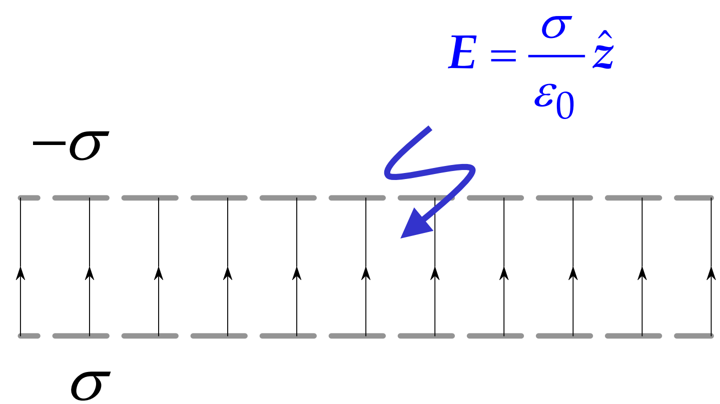
That is, just superpose the fields from two oppositely-charged planes. They add in between the planes, and cancel outside.

Application: ion drive

Even simple, uniform ***E*** can be quite useful, as in the **ion drive**, the latest wrinkle in spacecraft propulsion.

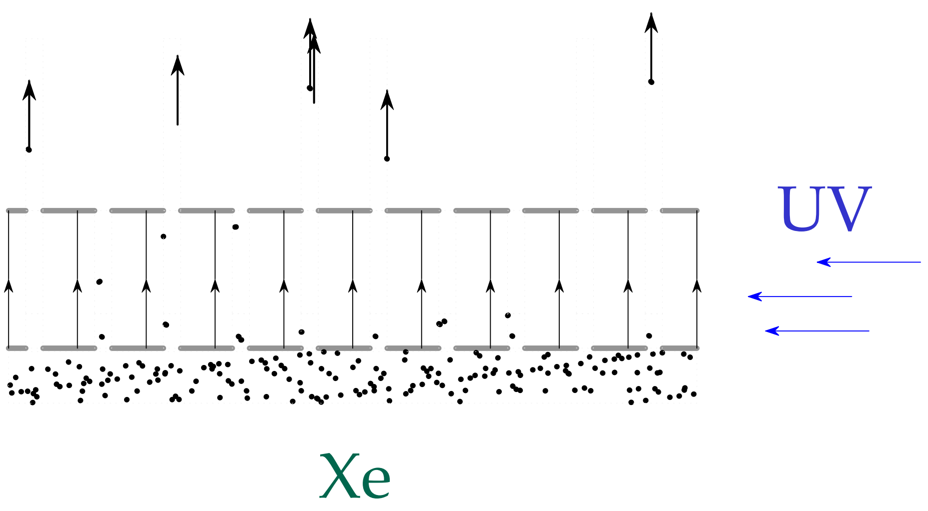
- Charge up two close parallel plates that have holes in them, much smaller than the spacing between the plates but large enough for atoms to leak through.
- As we just learned, the resulting ***E*** is uniform and perpendicular to the plates, and zero outside.





Ion drive (continued)

- Now let some heavy, easily-ionized gas, like **xenon**, leak in from the positively-charged side.
- Ionize the leaked atoms, e.g. with UV light.
- Then the xenon ions – positively charged – are accelerated by ***E***, and escape through the holes when they reach the other side.
    - The electrons from ionization move the opposite direction in ***E***.
- The larger is ***E***, the faster the ions are going when they escape.



Ion drive (continued)

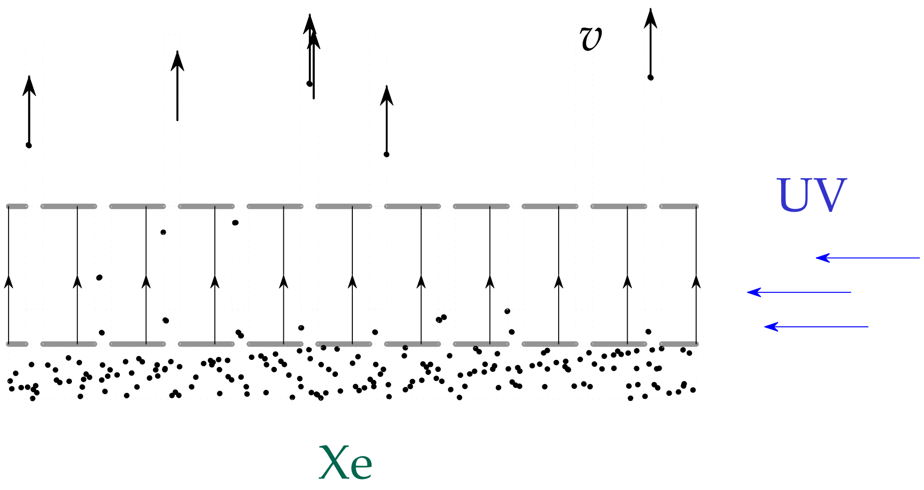
- The speed of the ions can easily be calculated from what you learned in PHY 121:

$$F = QE = ma \quad \Rightarrow \quad a = eE/m_{\mathrm{Xe}}\ ; \quad d = \frac{1}{2}at^2 \quad \Rightarrow \quad t = \sqrt{\frac{2dm_{\mathrm{Xe}}}{eE}}$$

$$v = at = \sqrt{\frac{2eEd}{m_{\mathrm{Xe}}}} \qquad m_{\mathrm{Xe}} = 2.2\times10^{-25}\,\mathrm{kg} \cong 124m_{\mathrm{H}}$$

- If N ions per unit time escape, they carry away momentum at the rate

$$\frac{d\boldsymbol{p}}{dt} = N\boldsymbol{v} = N\sqrt{\frac{2eEd}{m_{\mathrm{Xe}}}}\hat{\boldsymbol{z}}$$

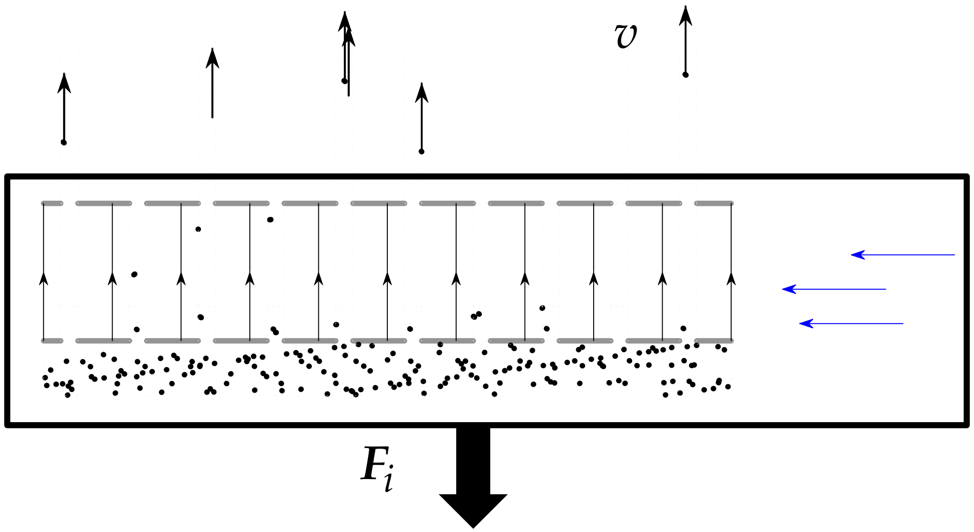


Ion drive (continued)

- But momentum is conserved, so the assembly containing the plates, gas supply, UV light, *etc*. experiences a **thrust**,

$$\boldsymbol{F}_i = -\frac{d\boldsymbol{p}}{dt} = -N\sqrt{\frac{2eEd}{m_{\text{Xe}}}}\hat{\boldsymbol{z}}$$

- …and thus the assembly (mass M) is propelled in the direction opposite that of the escaping ions, at a constant acceleration F_i/M which is larger, the larger is E.



Ion drive (continued)

- This is the principle behind NASA's NSTAR ion drives, which have been used on the *Deep Space 1* and *Dawn* missions to asteroids and comets 1.3-2.6 times further away from the Sun than Earth (r = 1.3-2.6 AU).
- The drives are powered by solar-panel generators.
- Typically NSTAR drives consume 2.3 kW of electrical power and produce 0.02 lb of thrust.
- Doesn't sound like much? Read on…

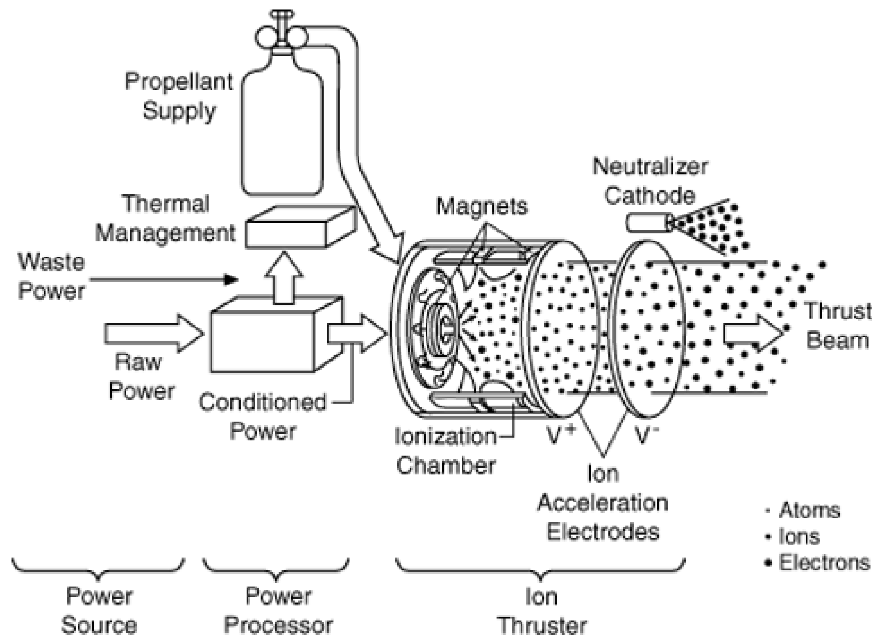
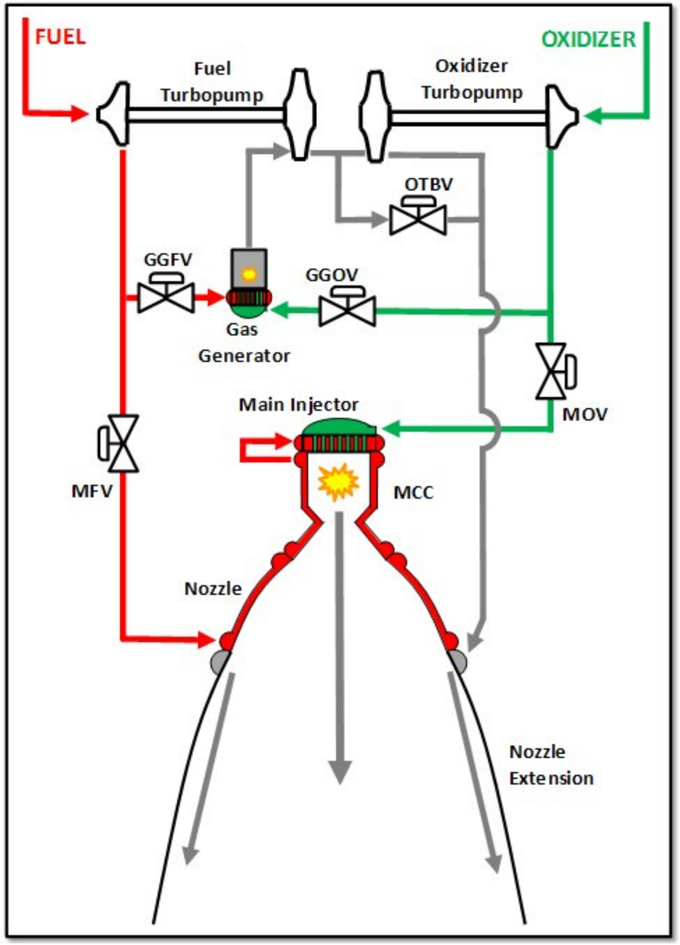


Diagram of *Dawn*'s ion drive (JPL/NASA).

Ion drive (continued)

- Ordinary rockets work by heating their fuel to high temperature and pressure, and letting some of this escape to produce thrust.
- Compared to ion drives, rockets are inefficient, since the ignited fuel atoms move in random directions in the ignition chamber.
- Thus an ion drive can reach higher speeds than a rocket of the same mass, though it may take longer to get to high speeds.



Schematic diagram of the J-2X rocket motor (NASA).

Ion drive vs. rockets: tale of the tape

Let's compare the best rocket with the best ion drive, for taking a 1000 kg satellite out of the Solar system.

- Best rocket: the mighty Saturn V, famous from the Apollo missions.
  - Total mass of satellite plus three-stage rocket $= 2.9 \times 10^6$ kg.
  - Initial thrust $= 3.9 \times 10^7$ N $= 8.7 \times 10^6$ lb. 1.7 x Falcon Heavy; 6 x Delta IV H; 40x Delta II H.
  - Also 1.2 x Energia; 2.8 x Ariane 5; 3.9 x Proton-M, 4.0 x Long March CZ-5, 7.6 x GSLV-III.
- Best ion drive: SAFE-400 nuclear reactor (100 kWe), 43 NSTAR drives; total mass 600 kg.

Launch of Apollo 11, 1969, on a Saturn V booster (NASA).

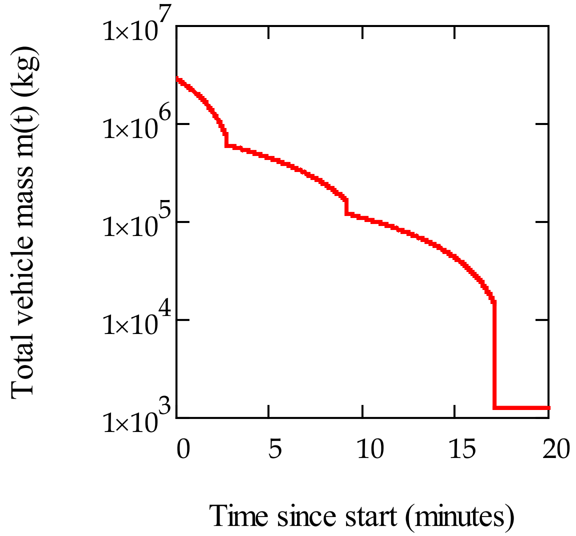
Ion drive vs. rockets: tale of the tape (continued)

Start the two vehicles with the same mass – adding Xe to the ion-drive vehicle til its mass is the same as the other – and accelerate as long as they are producing thrust. Their mass decreases as they go along, and use up fuel, so the acceleration is a function of time:

$$a(t) = F_{\text{thrust}}/m(t)$$

$$v(t) = \int_0^t a(t')\,dt'$$

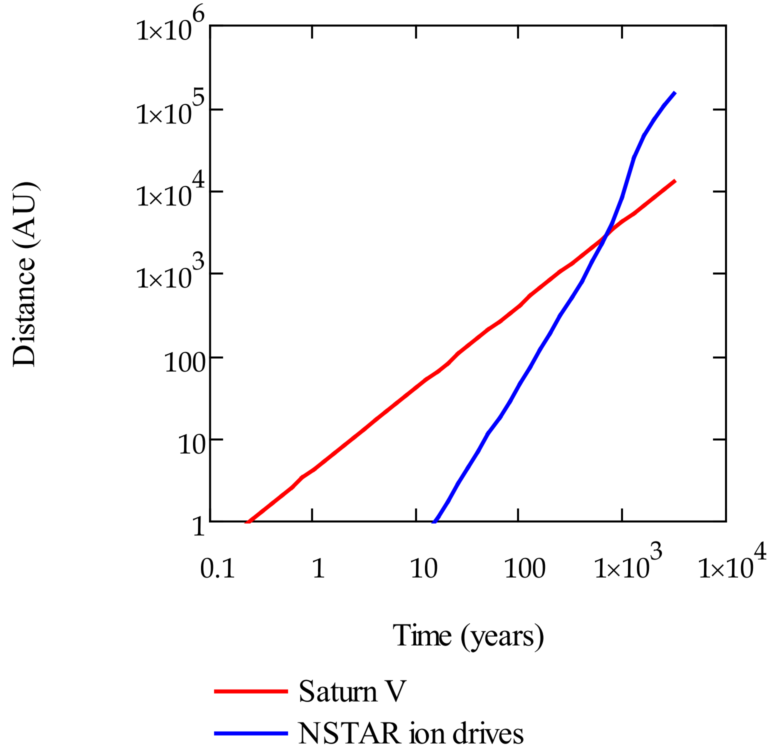
$$x(t) = \int_0^t v(t')\,dt'$$

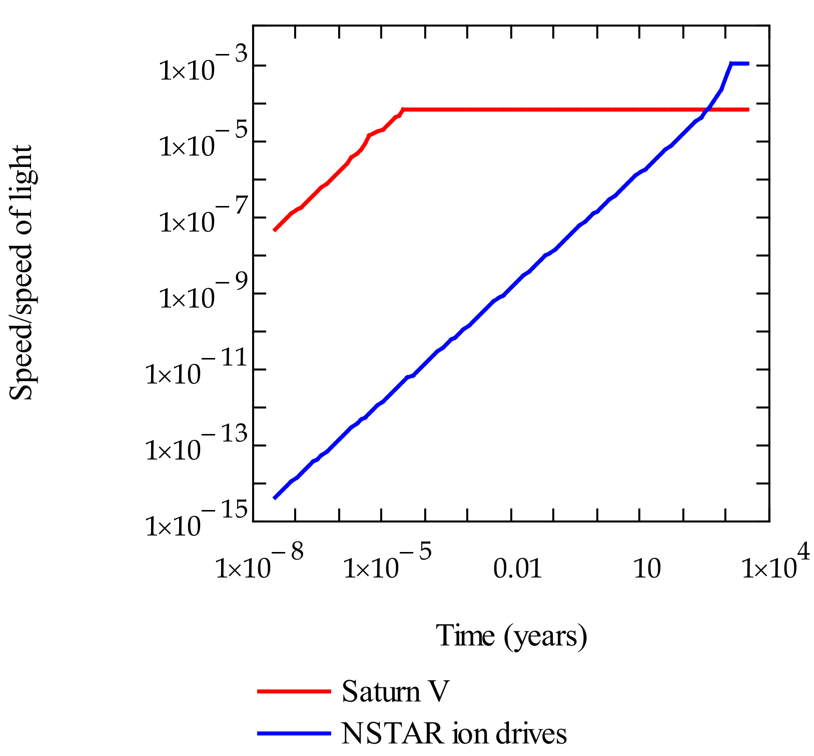


Mass of the rocket-powered vehicle, as the Saturn V expends and ejects its three stages.

Ion drive vs. rockets: tale of the tape (continued)

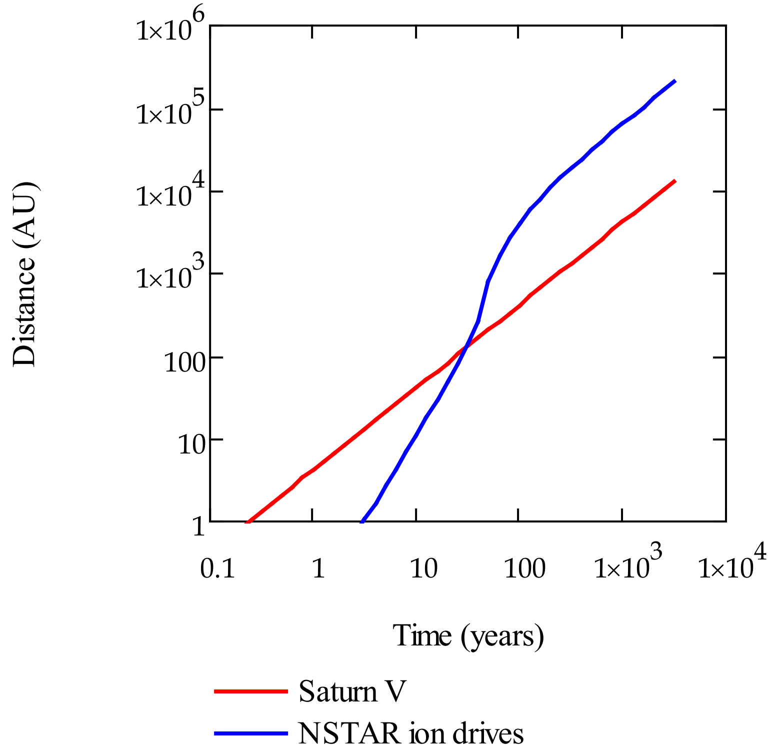
Result: the Saturn V can get anywhere within the Solar system faster than *current* NSTAR ion drives, but eventually the ion drive goes farther and flies faster.

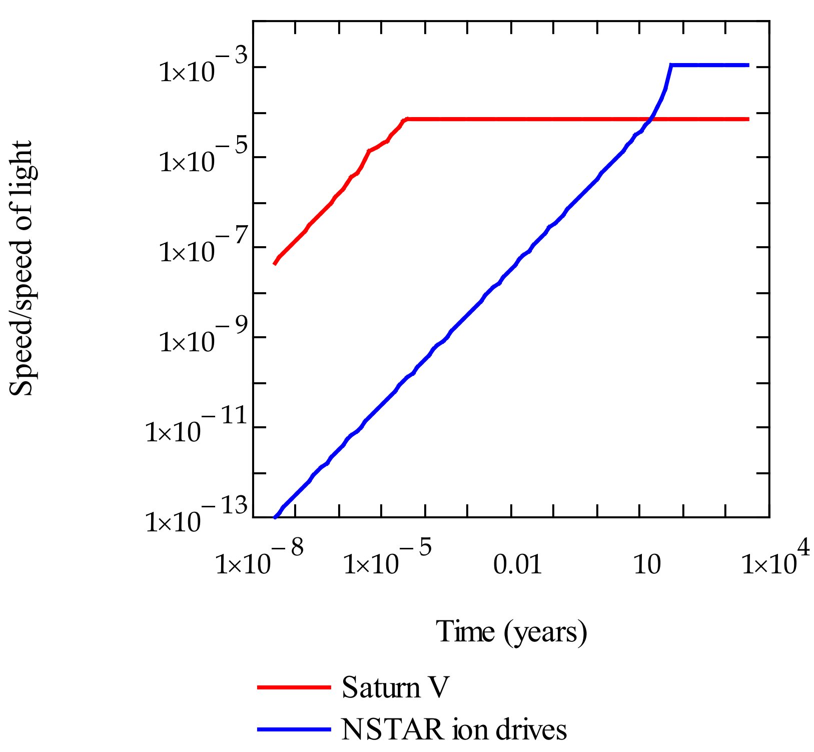




Ion drive vs. rockets: tale of the tape (continued)

Ion drives have much more room for improvement than rockets. Here are the results for NSTAR drives that can run on 100 W of electrical power:





Ion drive vs. rockets: tale of the tape (continued)

If in addition E could be made 10000 times larger, the ion drive's top speed begins to look respectable, at 10% the speed of light, and gets to the nearest stars in about 1500 years.