Error Analysis and Statistics for Students in Introductory Physics Courses

(A lot of material in one lecture)

It is the only lab lecture in the course.

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Reference: ***Physics Introductory Lab Manual Appendix A*****.**
Located at

http://web.pas.rochester.edu/~physlabs

And also **Review Module 1 - located on Blackboard**

Error analysis is probably the most important topic for scientists and engineers.

But it is not covered in any other course.

Over the years, students who have done research in engineering have typically used the PHY121 lab manual when they had to quantify the uncertainty in the results of their research.

Is statistics relevant to you personally?
Consider the last election

	Month 1	Month 2
McCain	42%	41%
Obama	40%	43%
Undecided	18%	16%

$\pm 4\%$

HOW DO WE DETERMINE THIS UNCERTAINTY??

AND WHAT DOES IT MEAN??

Headline: Obama surges past McCain in polls!

Poisson Statistics: **The error in counting a Number which is expected to have a value of N**

Poisson Statistics is the only case for which we can get the error from a single measurement. In all other cases we need to have more than one measurement.

For 1000 counts the error is √(1000) or about 30 (3% error)

This is why polls use ask about 2000 people who the plan to vote for.
If half (1000) say Obama, the sampling error is 3%
If we asked 20 people, **if half (10) would say Obama: the error is sqrt (10) or about 3. (30% error) - Not enough**

The standard statistical error in a counting experiment is:

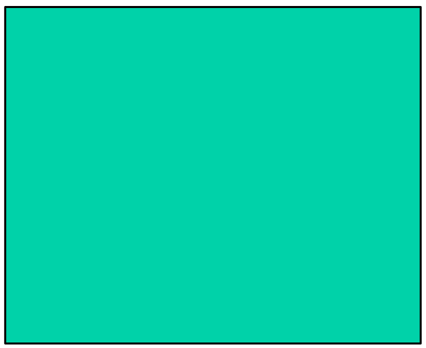
$$\Rightarrow \sigma = \sqrt{N}$$

This is called Poisson Statistics.
Here The fractional error is:

$$\frac{\sigma}{N} = \frac{1}{\sqrt{N}}$$

Measuring number of blades of grass N in a big lawn

Measure n=104 blades in a small sample of area= a_sample



Measure area of entire lawn area = A_lawn

$$N = n * [A\_lawn/a\_sample] \pm \Delta N$$

$$\Delta N = \sqrt{(n)} * [A\_lawn/a\_sample]$$ = **statistical error in N**

about 10% ($\sqrt{(104)}$ is about 10)

What are the systematic errors (a possible bias) in N ?

Systematic errors (a possible bias) in N

1. **Systematic error in measuring the area of the sample square (can be estimated by how we measured it, e.g. ruler)**

2. **Systematic error in measuring the area of the lawn (can be estimated by how we measured it – e.g. tape measure)**

1. **How representative of the lawn is the sample area that we chose to measure → systematic error which harder to estimate → How do we estimate it?**

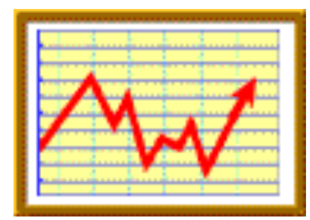
How representative of the lawn is the sample area ?

Need to take more samples in different areas of the lawn to find the difference between different sections

Is error analysis and statistics relevant to you personally?

Global Warming is it real? If so, how much is from human activity?

Medical Research and risks



Effect of EM radiation; Is it detrimental?, should I use a headset with my cell phone? Or stick to speaker phone?

What is wrong with this?

$$2\frac{1}{2}$$

A quantitative measurement is meaningless without specifying the units and error.

A quantitative measurement is described by a VALUE, an ERROR and UNITS.

What about qualitative statements?

(e.g. far or near, big or small)

We use ***order of magnitude estimate*** to describe what other people would call a "qualitative statement". Order of magnitude means that we know it within a factor of 10.

So in Physics, even qualitative statements are quantified (it just means that the error is in the exponent (which we know to plus or minus 1)

i.e. Syracuse is about 100 miles from Rochester. (It is not 1000 miles and it is not 10 miles).

Accuracy:

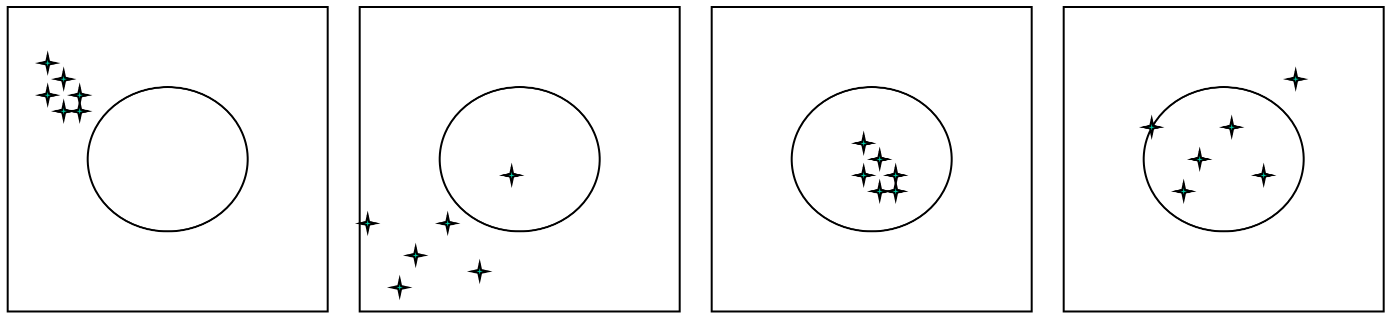
A measure of closeness to the "truth". - We will call these "systematic errors" It is the most difficult error to estimate since we do not know the true value. For some PhD theses, it takes a year to get a reliable estimate of the systematic uncertainty

Precision:

A measure of reproducibility. - We will call these "random" or "statistical" errors (uncertainties)

$$X = 10 \pm 2 \text{ (stat)} \pm 3 \text{ (sys)}$$

Accuracy vs. precision



Accurate	✗	✗	✓	✓
Precise	✓	✗	✓	✗

Accurate = small systematic error

Precise = small statistical (random) error -> Reproducible

Summary: Types of errors

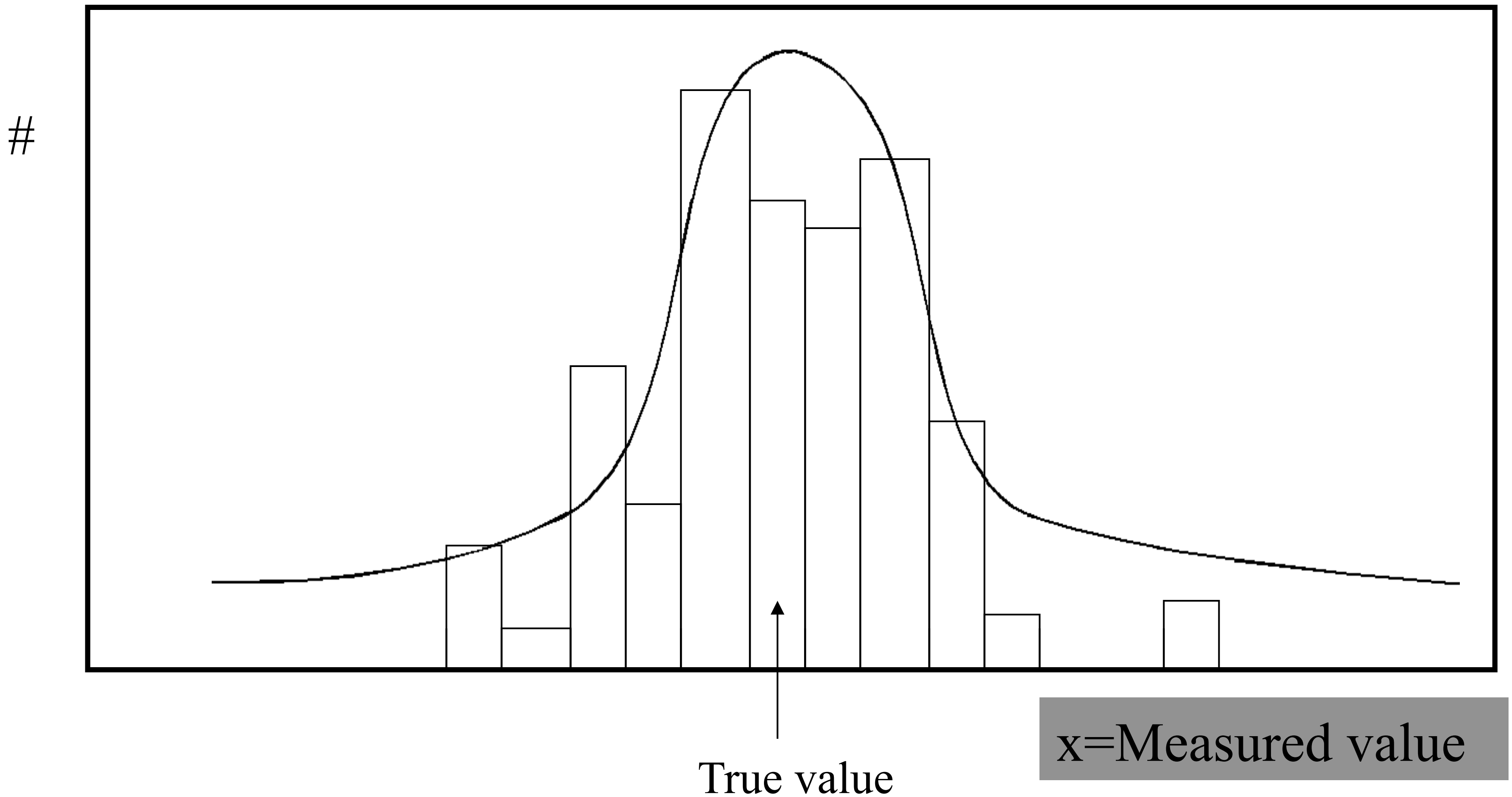
Statistical error: Results from a random fluctuation in the process of measurement. Often quantifiable in terms of "number of measurements or trials". Tends to make measurements less *precise*.

Systematic error: Results from a bias in the observation due to observing conditions or apparatus or technique or analysis. Tend to make measurements less *accurate*. *Example: Effects that were not thought of at the time of the experiment, but bias the result.*

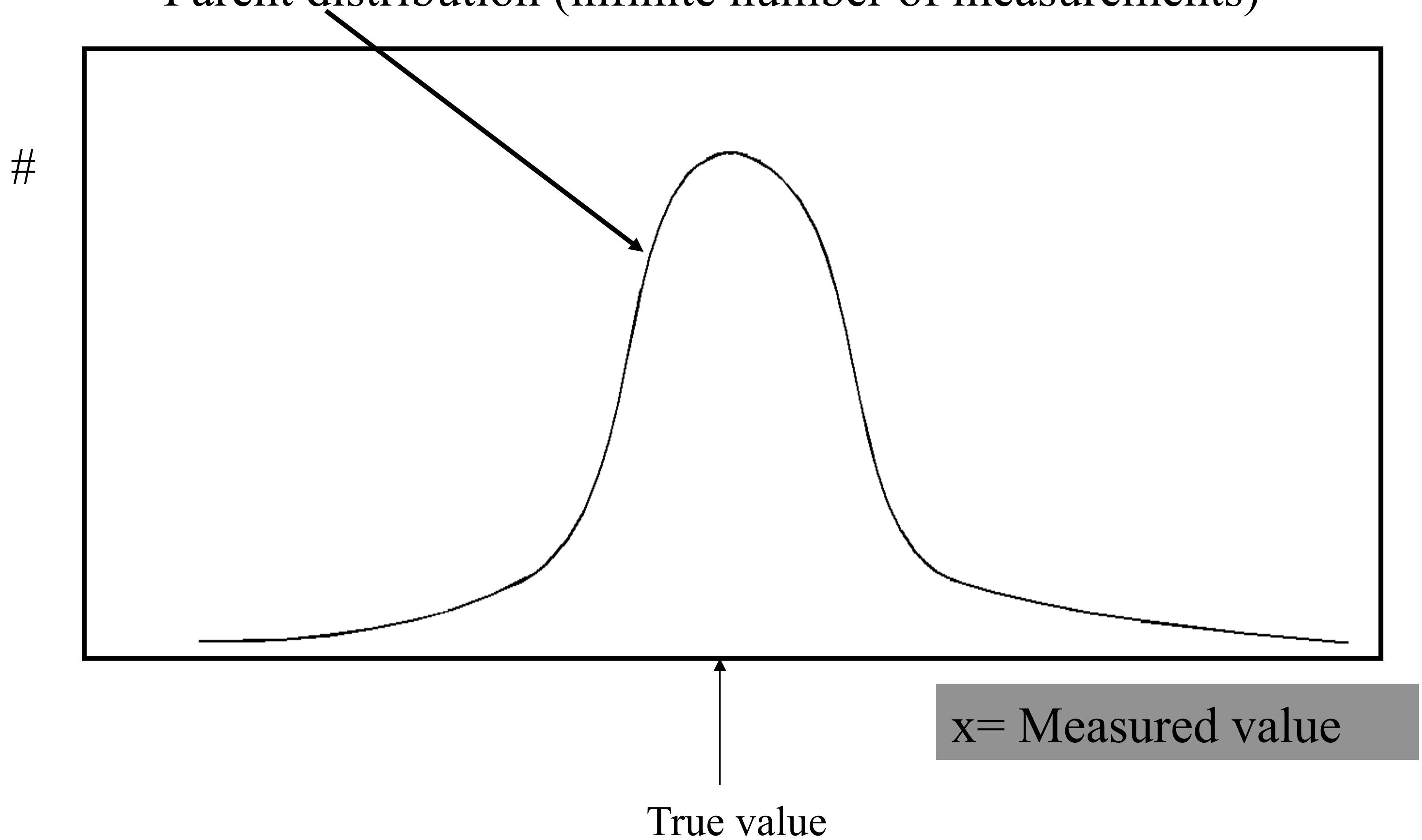
(e.g. to predict the presidential election, we polled only students from the University of Rochester)

Example 2: Lets do another experiment to illustrate random and systematic errors. E.g "a measurement of the acceleration of gravity g"

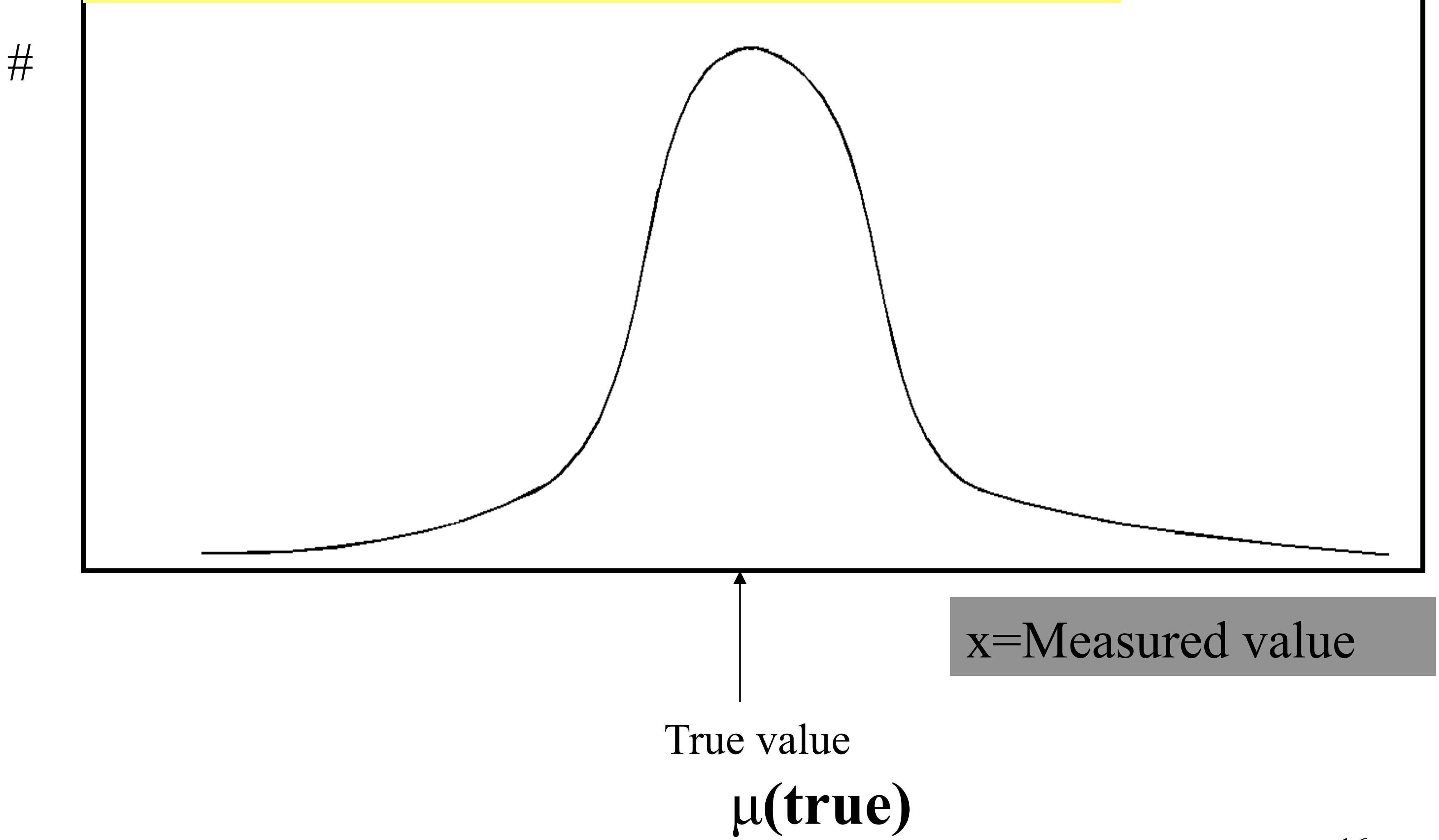
Statistical error can be determined from multiple measurements: A finite number of multiple measurements



Parent distribution (infinite number of measurements)

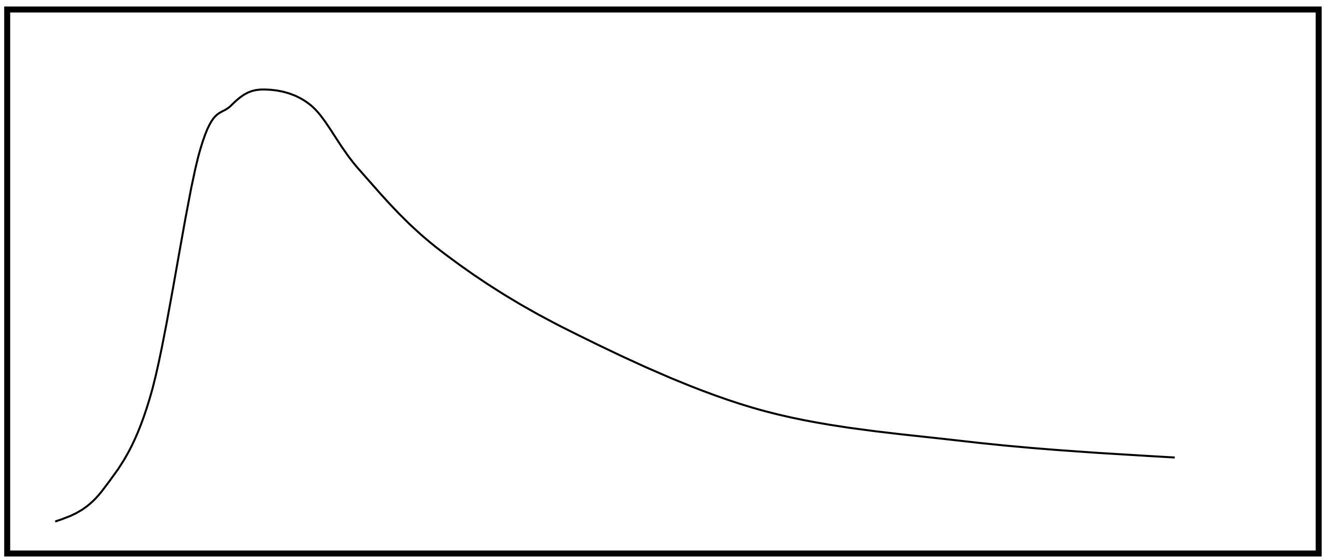


The game: From N (not infinite) observations, determine best estimate of "μ" and the "error on μ" … without knowledge of the "truth".



The parent distribution can take different shapes, depending on the nature of the measurement.

The two most common distributions one sees are the **Gaussian** (normal) and **Poisson** distributions.

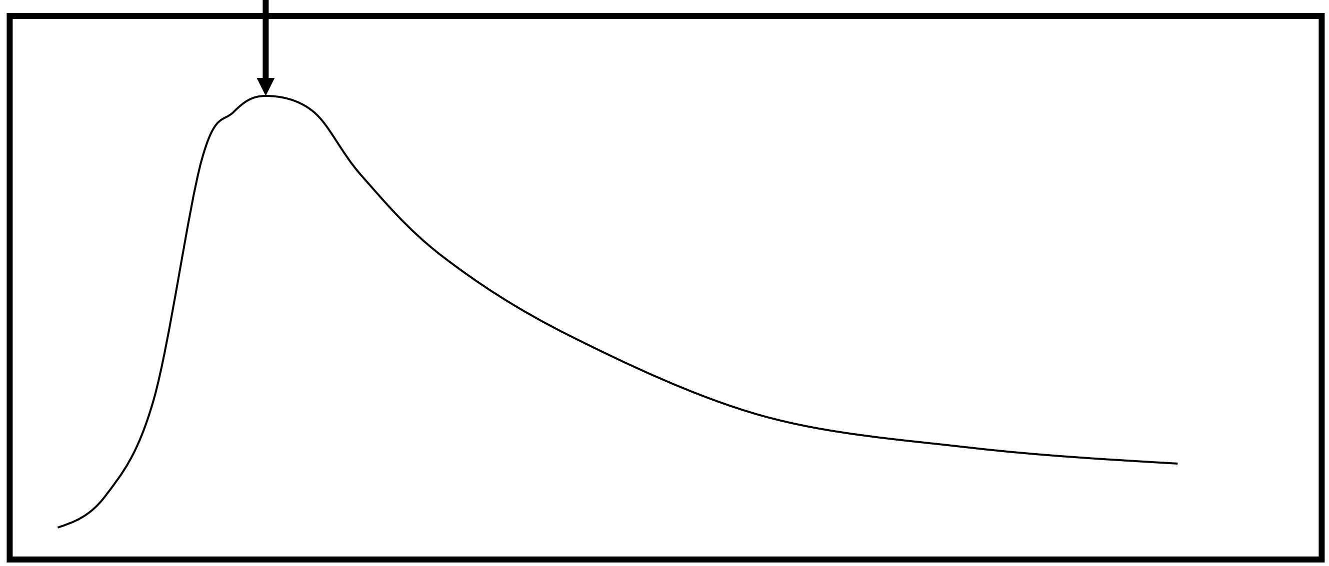


Example of a probability distribution

Most probable value (mode)

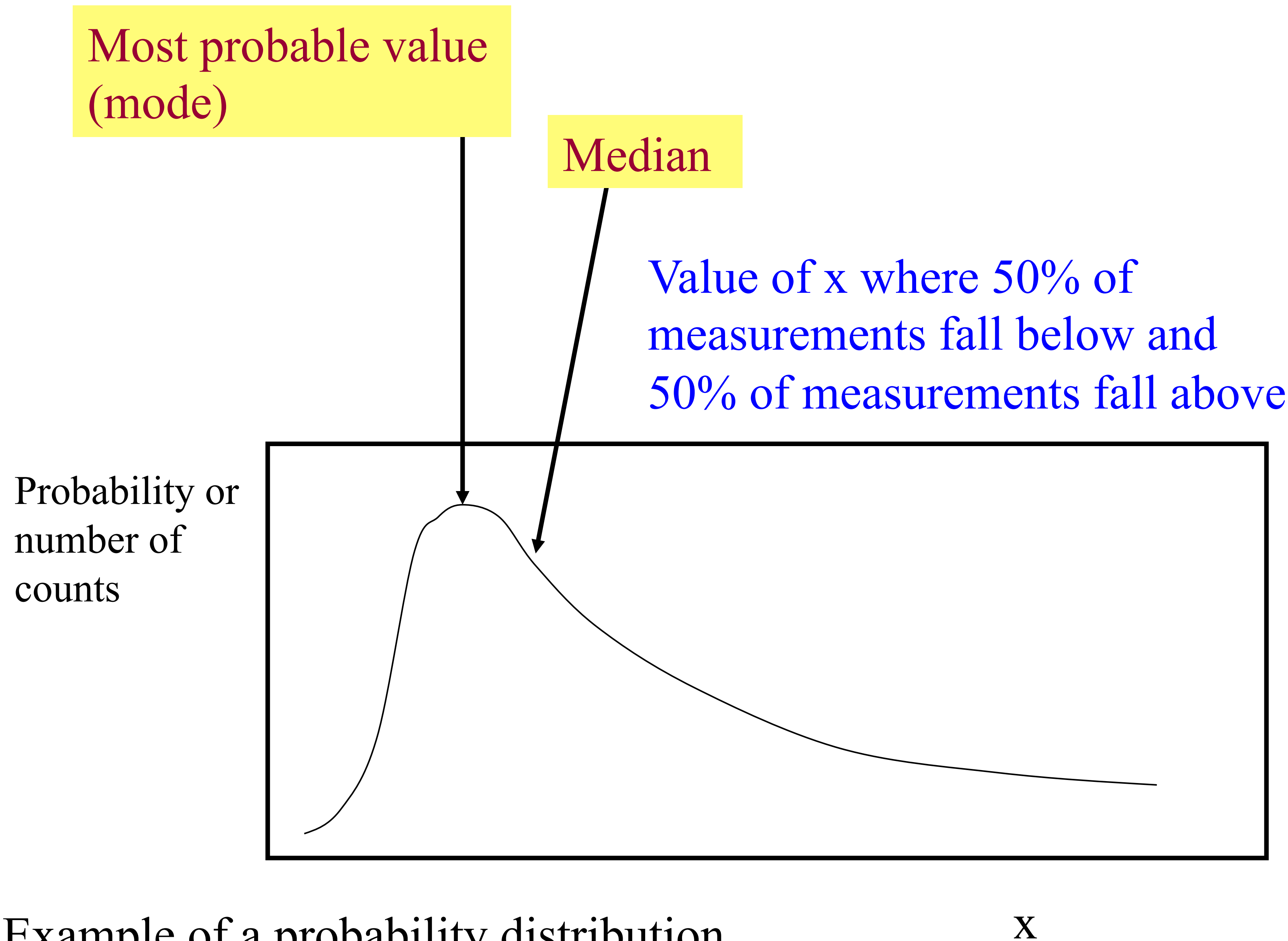
Highest on the curve. Most likely to show up in an experiment.

Probability or number of counts

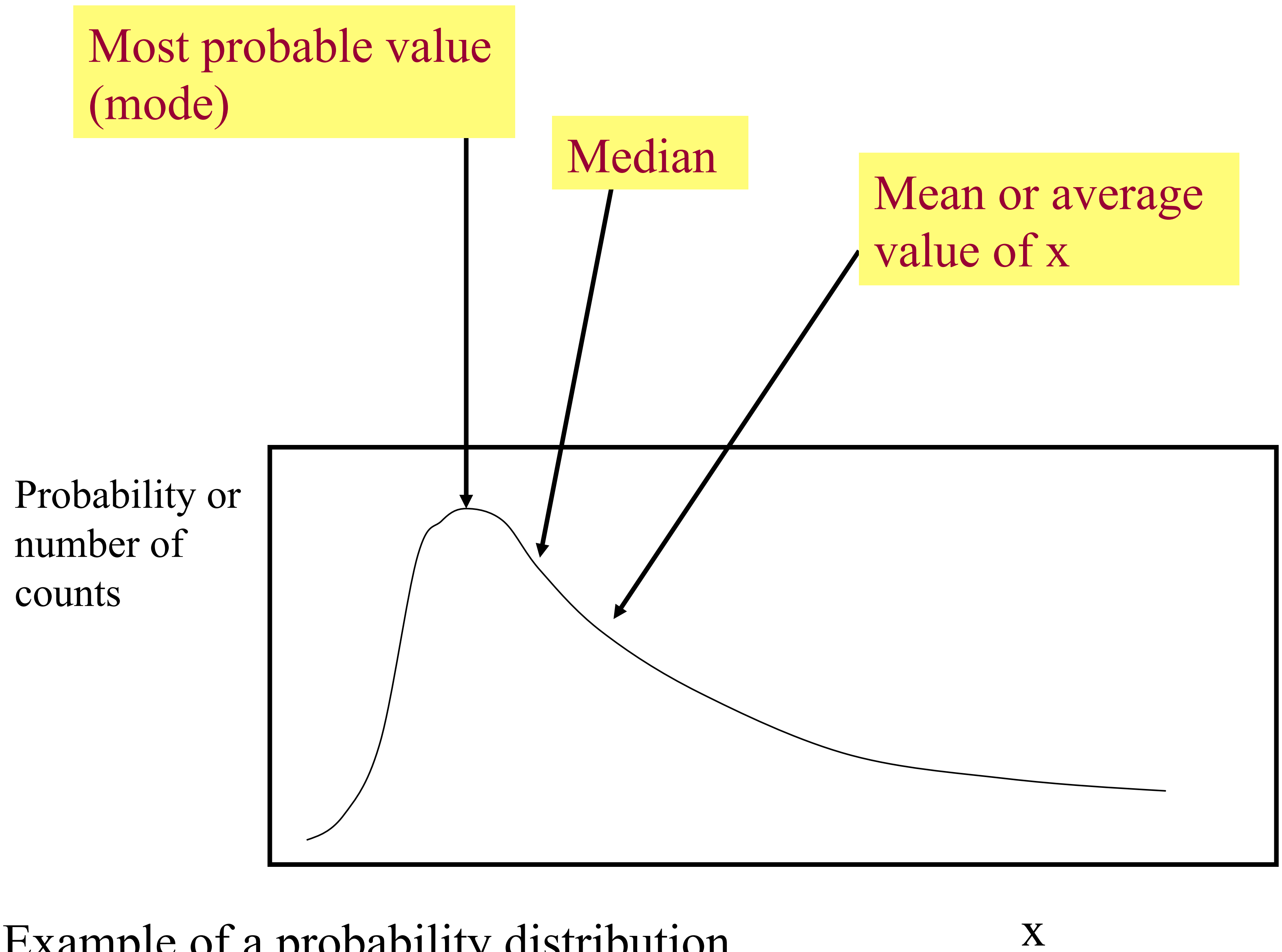


x

Example of a probability distribution

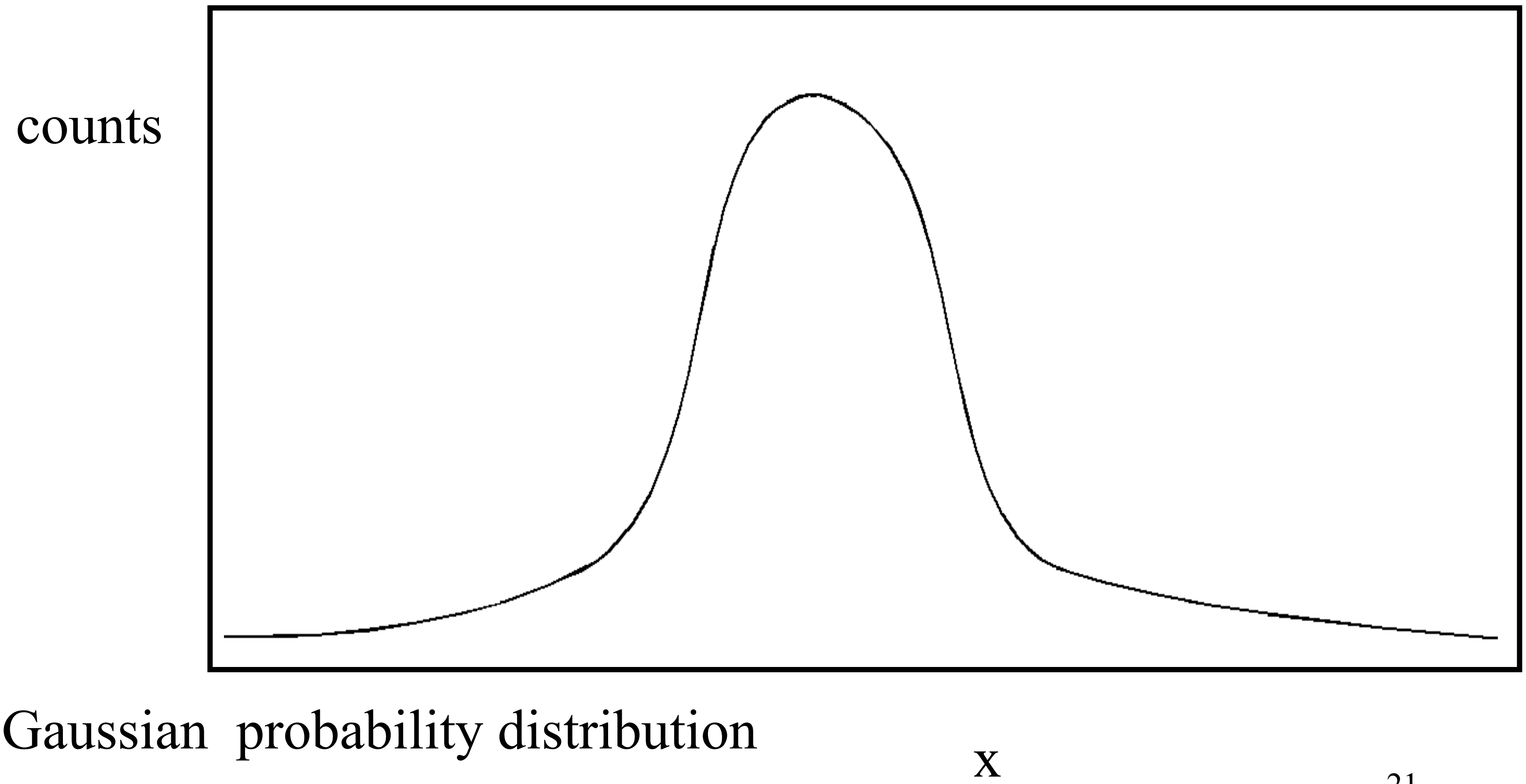


Example of a probability distribution



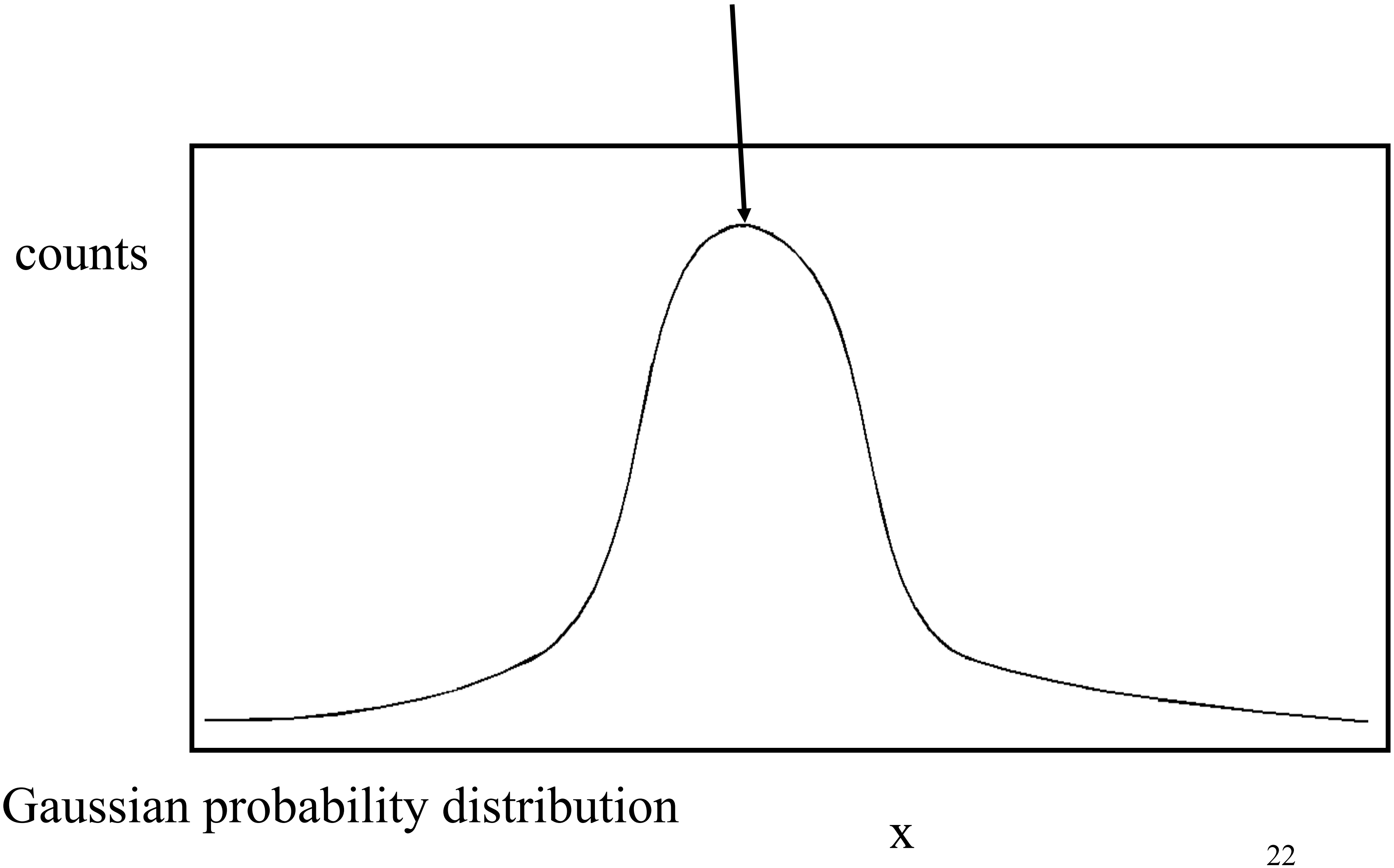
Example of a probability distribution

The most common distribution one sees (and that which is best for guiding intuition) is the **Gaussian** (Normal, Bell shaped) distribution.



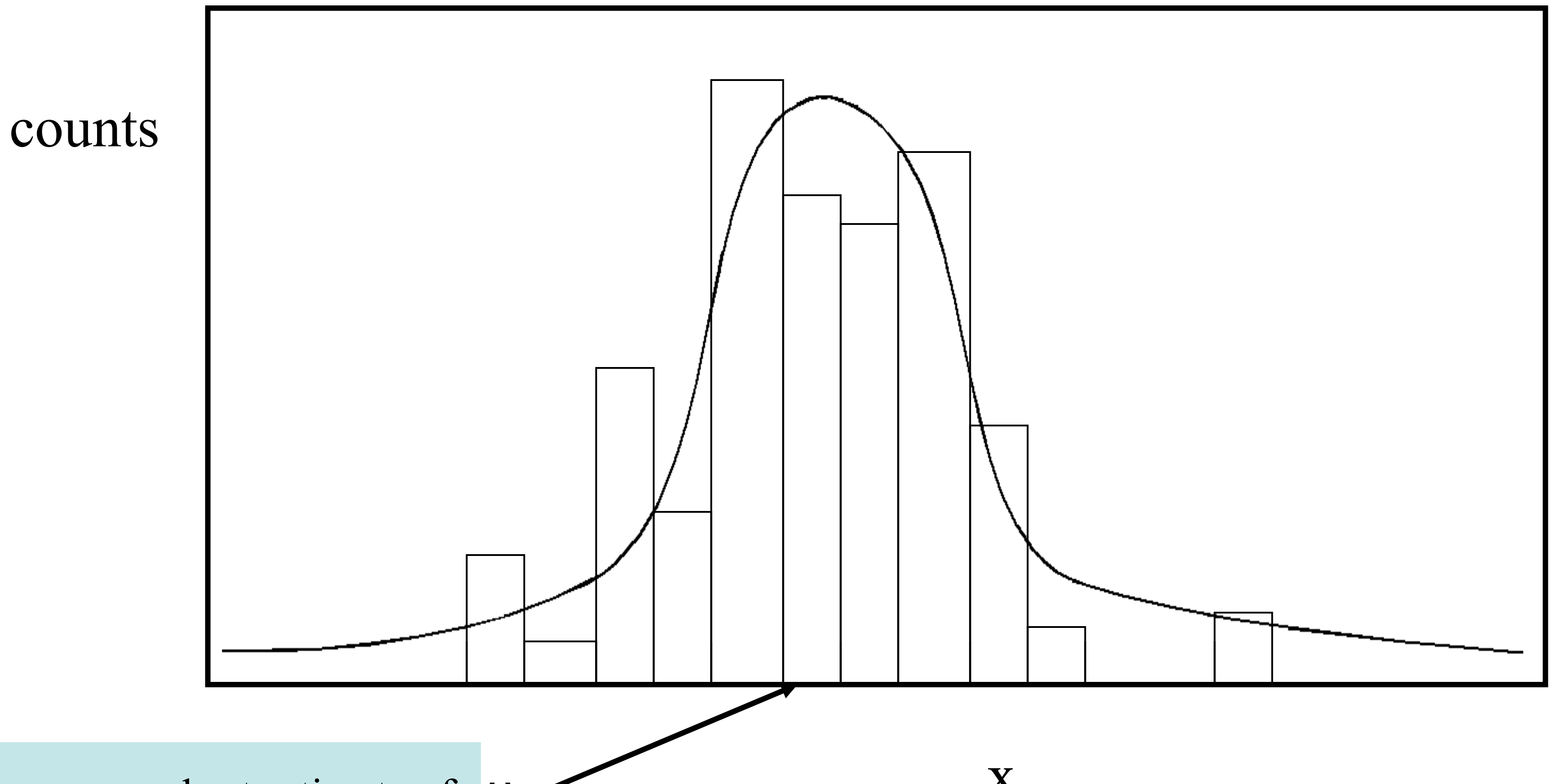
Gaussian probability distribution

For this distribution, the most probable value, the median value and the average are all the same due to symmetry.



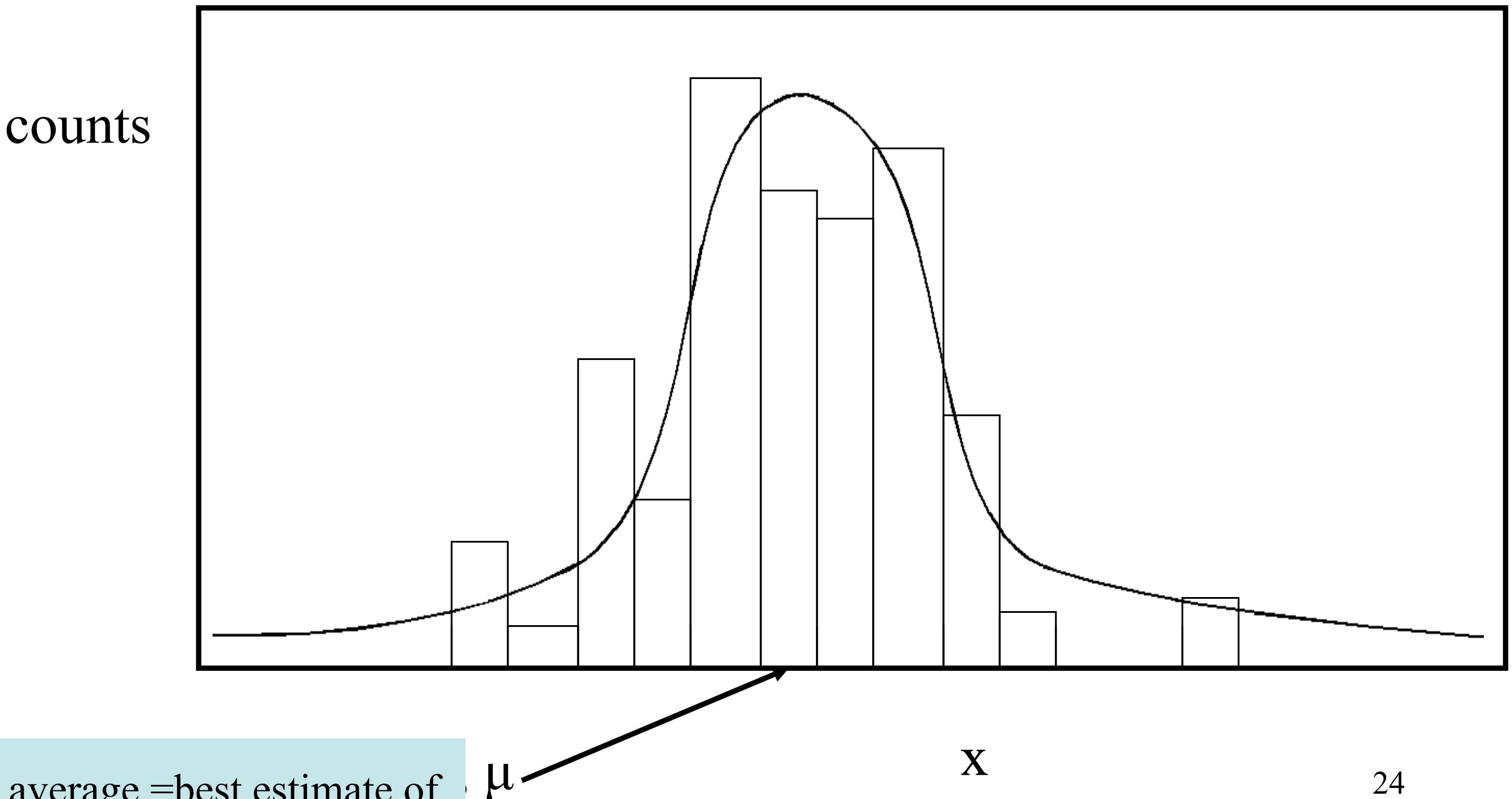
Gaussian probability distribution

The most probable estimate of μ (the true mean) is given by the mean (=average) of the distribution of the N observations



Mean= average =best estimate of μ

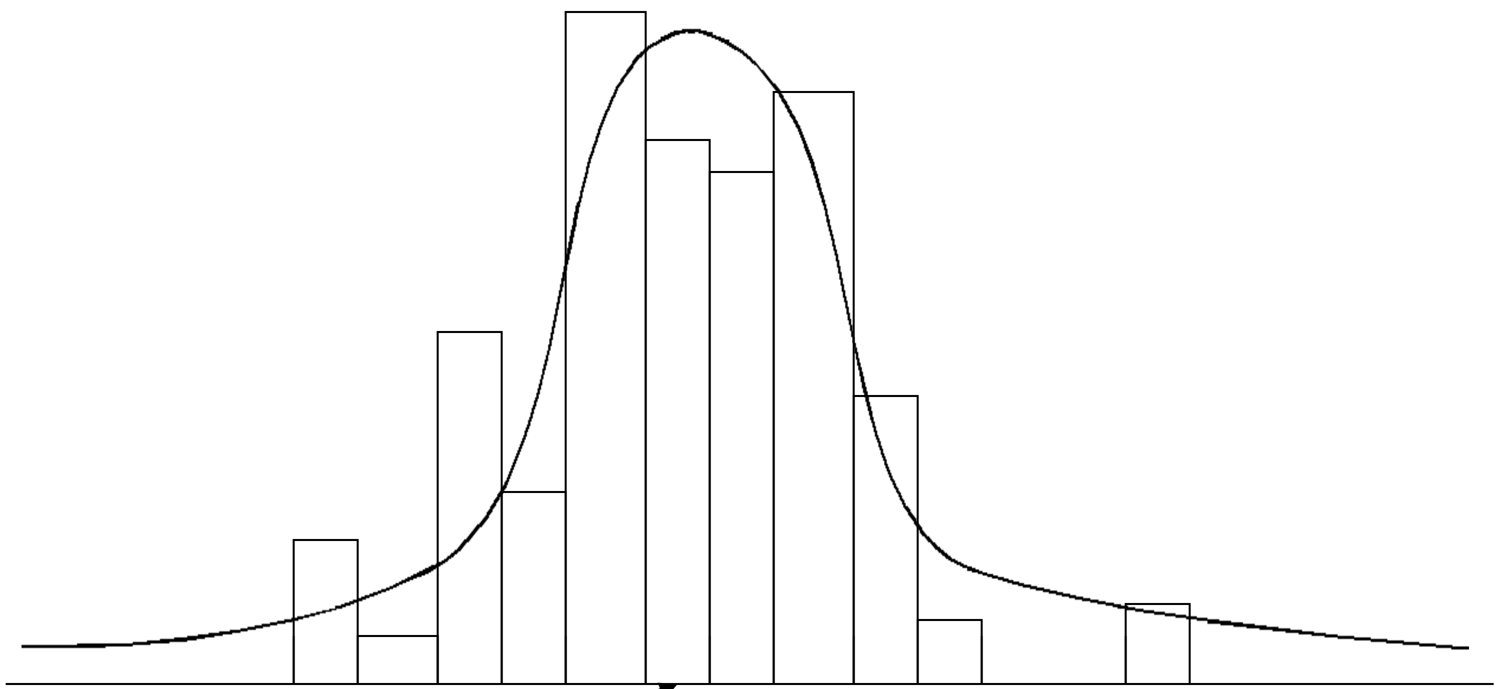
$$"\mu" = \bar{x} = \frac{x_1 + x_2 + \cdots + x_{N-1} + x_N}{N} = \frac{\sum_{i=1}^{N} x_i}{N}$$



Mean= average =best estimate of μ

OK we can get the best estimate for the true value by taking the average of our measurements

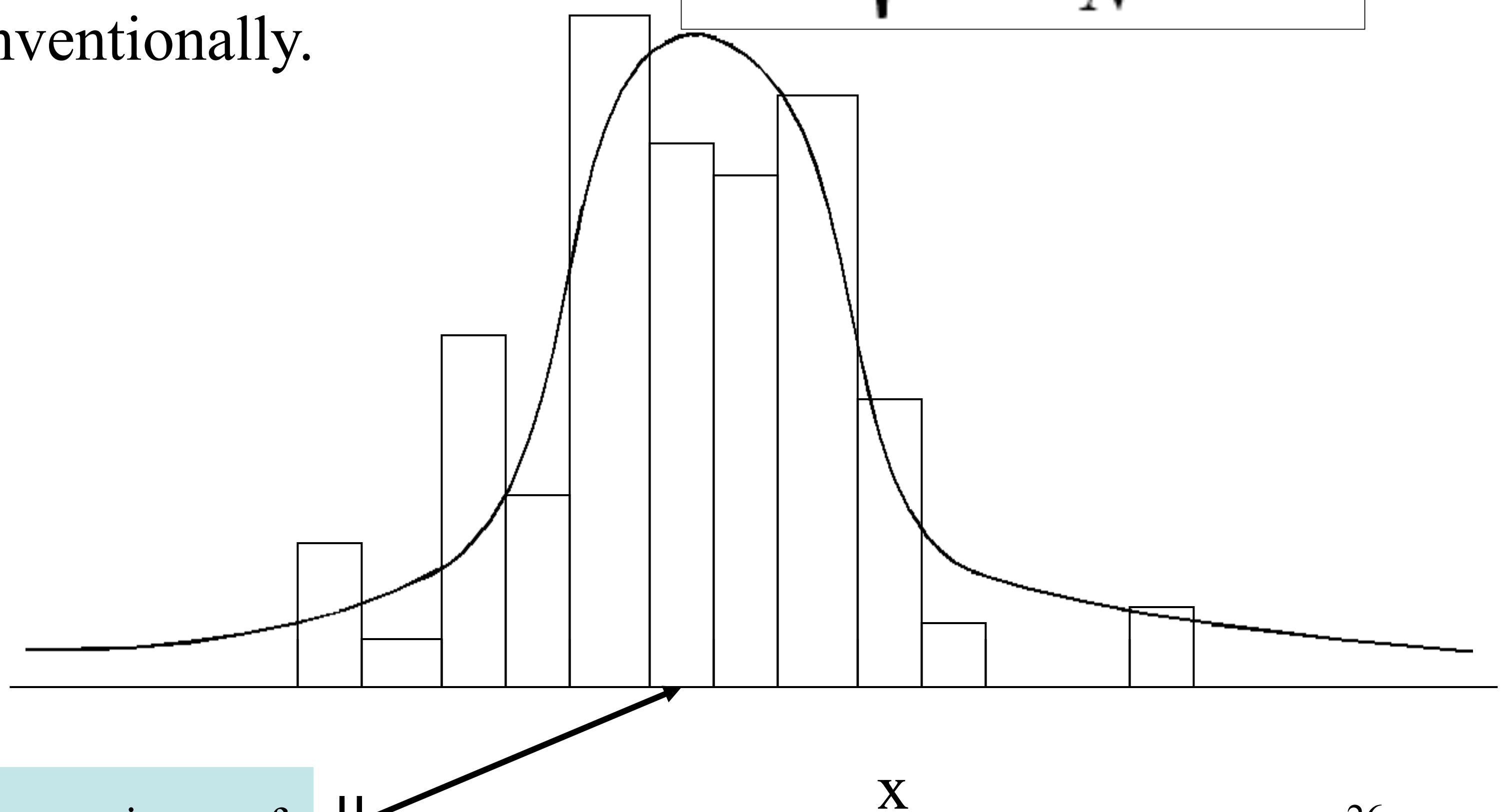
How do we get the error?



Mean= average =best estimate of μ

The “standard deviation” is a measure of the error in each of the N measurements.. This is the Standard Error that is used conventionally.

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}}$$



Mean= average =best estimate of μ

μ(true)is unknown. So we use the measured mean $\overline{x}$ (which is the best estimate of the true mean μ).

Change denominator to increase error slightly due to having used the measured mean (which is only an estimate of the true mean μ)

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \overline{x})^2}{N-1}}$$

This is the form of the best estimate of the true standard deviation that you use in practice. It is the best estimate of the error in a single measurement of x.

This quantity cannot be determined from a single measurement. One must have several measurements (except for counting experiments)

Simplest example N=2 measurements: X1= 9 and X2=11.

Mean = Average = best estimate of the true value

X(average) = (9+11)/2 = 10

Best estimate of the error in each of the two measurements

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \overline{x})^2}{N-1}}$$

$$= \sqrt{\frac{(9-10)^2+(11-10)^2}{(2-1)}} = \sqrt{2} = 1.4$$

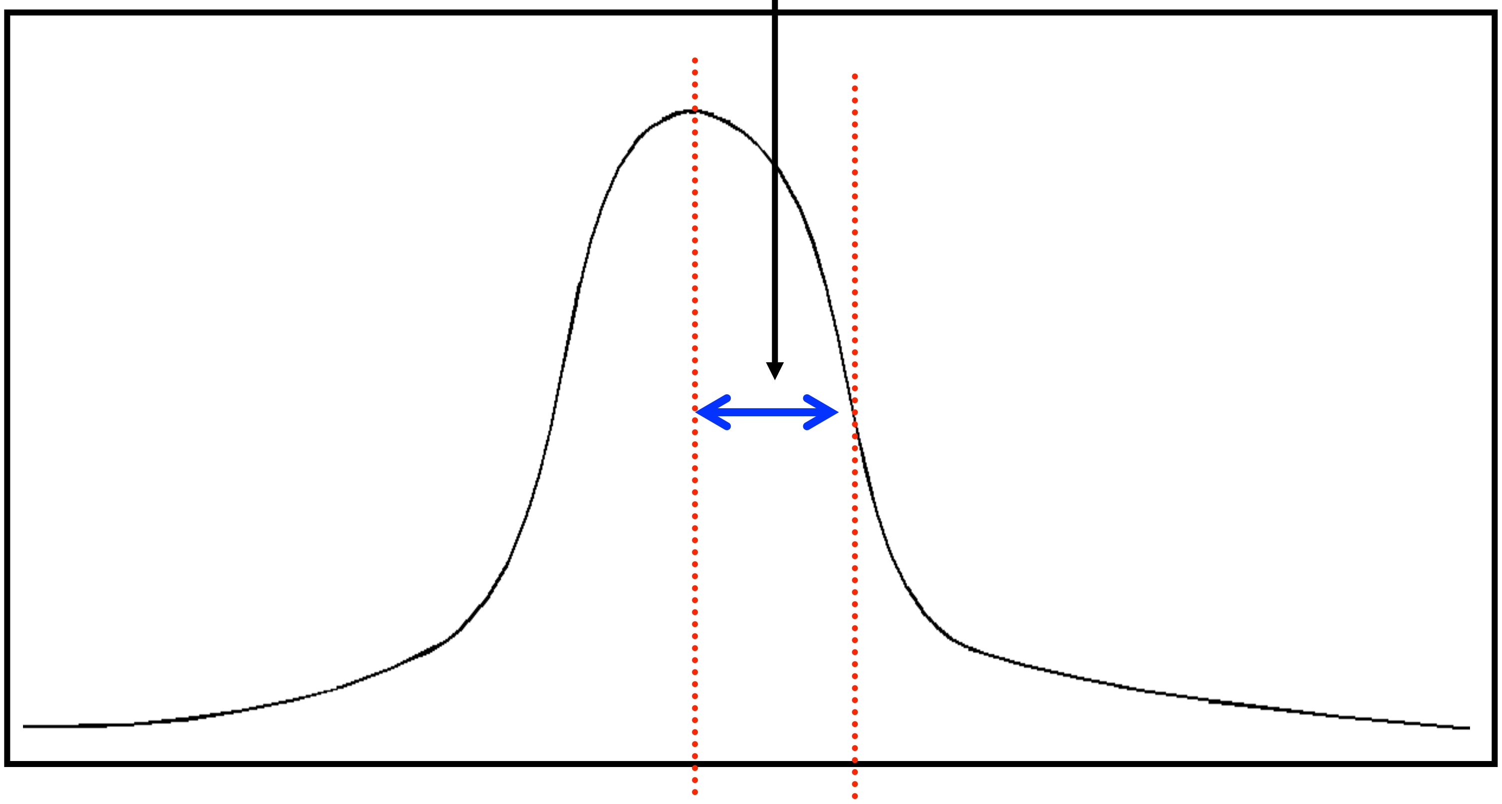
so X1= 9 $\pm$ 1.4 and X2= 11 $\pm$ 1.4

Gaussian (normal) distribution

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$$

1σ is roughly half width at half max

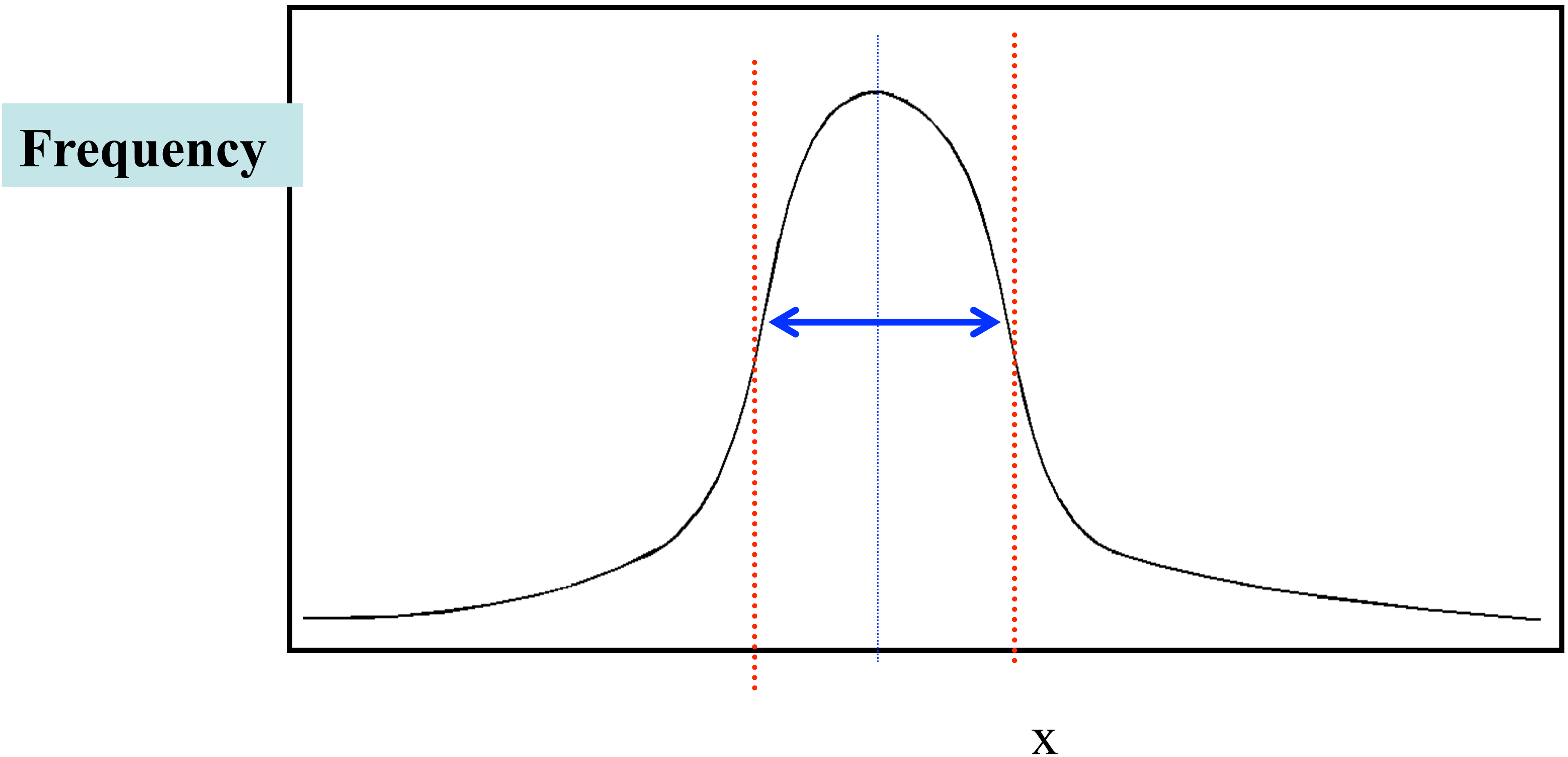
Frequency



Gaussian (normal) distribution

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$$

Probability of a measurement falling within ±1σ of the mean is 0.683

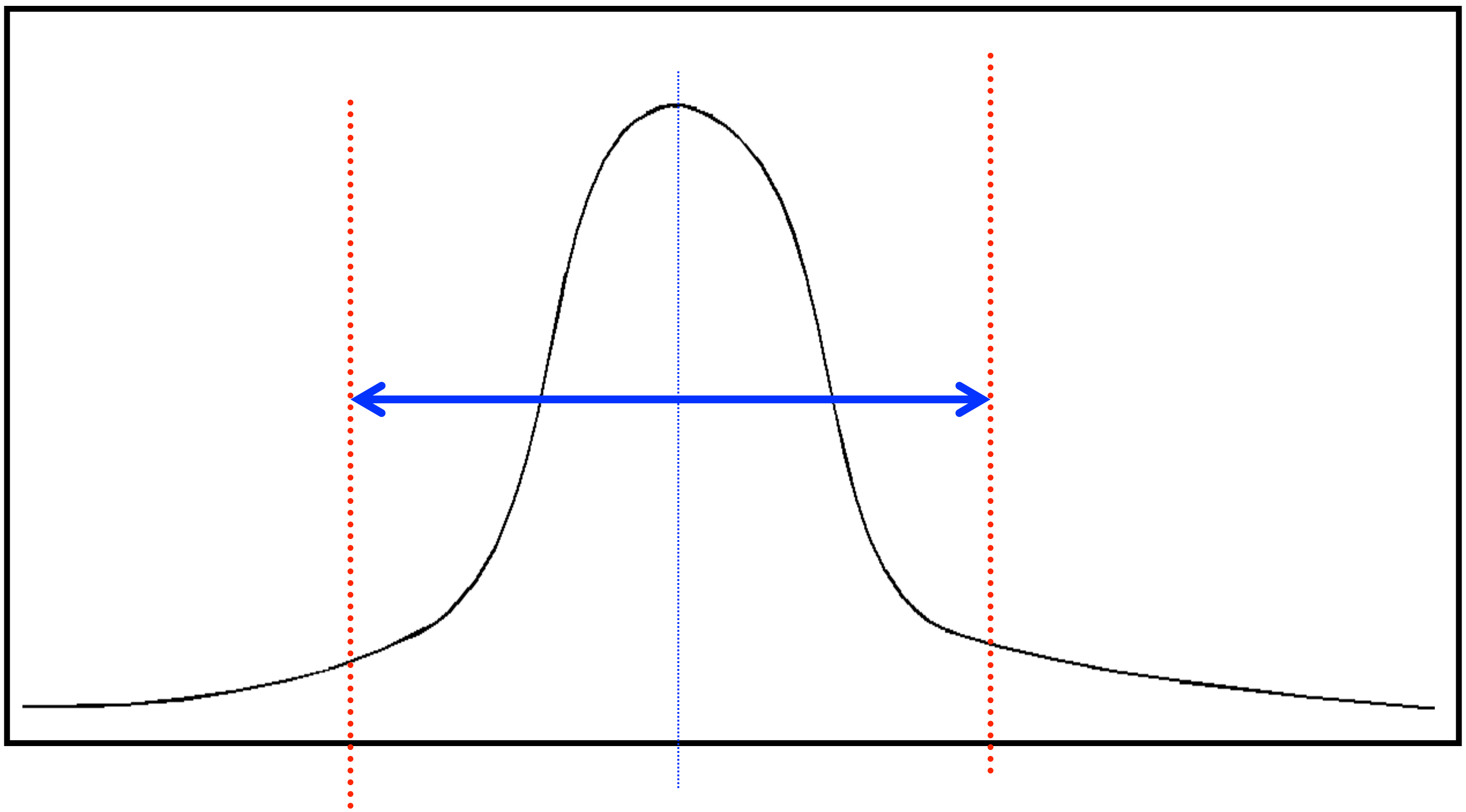


Gaussian (normal) distribution

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$$

Probability of a measurement falling within ±2σ of the mean is 0.954

Frequency



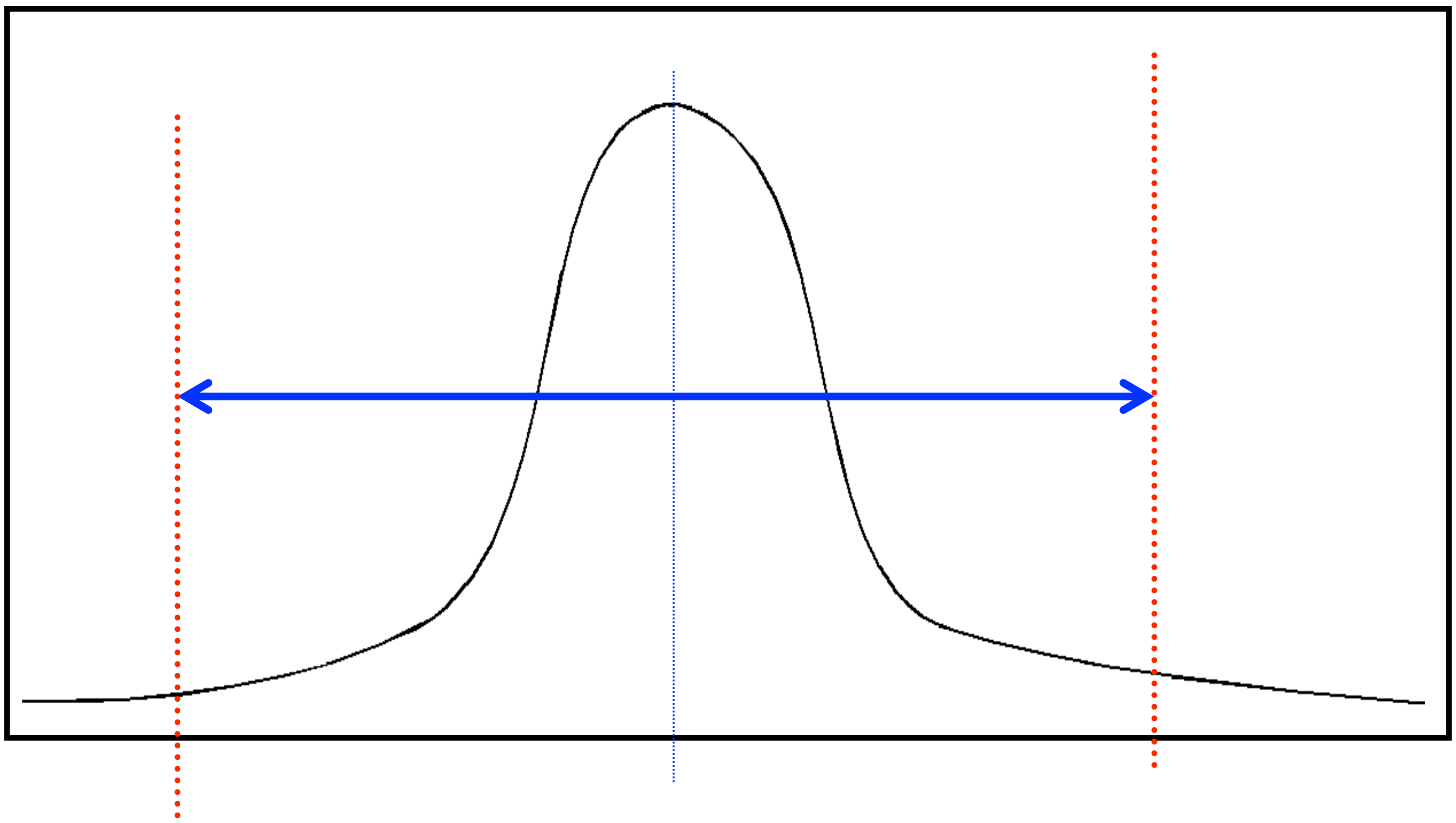
x

Gaussian (normal) distribution intuition

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$$

Probability of a measurement falling within ±3σ of the mean is 0.997

Frequency



(43 $\pm$ 4) % is not statistically different from (41 $\pm$ 4) %

	Month 1	Month 2	
McCain	42%	41%	
Obama	40%	43%	
Undecided	18%	16%	4%

Headline: Obama surges past McCain in polls!

The standard deviation is a measure of the **error** made in each **individual measurement**.

Often you want to measure the **mean** and the

[**error in the mean]** of several measurements

The average of several measurement should have a **smaller error** than the error in an individual measurement x.

Error in the mean

$$\sigma_m = \frac{\sigma_x}{\sqrt{N}}$$

Previous example N=2 measurements: \

X1= 9+- 1.4 and X2=11+- 1.4

Mean = Average of the two = best estimate of the true value

$$\bar{X}\text{ (average)} = (9+11)/2 = 10 \pm 1.4/\sqrt{2} = 1$$

Best estimate of the error in each of the two measurements

$$\sigma_m = \frac{\sigma_x}{\sqrt{N}}$$

Error in the average of 2 measurements is smaller than the error in each measurement

Another Numerical example:

I dropped a ball and 5 students used stop watches to see how long it took to hit the ground. Each student used their measurement to determine the acceleration of gravity g.

Using the measurements of the 5 students, what is the best estimate of g and its error?

Student 1: 9.0 m/s^2 Student 2: 8.8 m/s^2

Student 3: 9.1 m/s^2 Student 4: 8.9 m/s^2

Student 5: 9.1 m/s^2

We use this formula

$$"\mu" = \bar{x} = \frac{x_1 + x_2 + \cdots + x_{N-1} + x_N}{N} = \frac{\sum_{i=1}^{N} x_i}{N}$$

First get the mean (average)

N=5

Student 1: 9.0 m/s^2

Student 2: 8.8 m/s^2

Student 3: 9.1 m/s^2

Student 4: 8.9 m/s^2

Student 5: 9.1 m/s^2

$$g = \frac{9.0 + 8.8 + 9.1 + 8.9 + 9.1}{5} = 9.0\frac{m}{s^2}$$

Mean is 9.0 m/s^2 - Now find the error in each measurement of g

Student 1: 9.0 m/s^2

Student 2: 8.8 m/s^2

Student 3: 9.1 m/s^2

Student 4: 8.9 m/s^2

Student 5: 9.1 m/s^2

We use this formula

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1}}$$

$$\sigma_x = \sqrt{\frac{(9.0-9.0)^2 + (8.8-9.0)^2 + (9.1-9.0)^2 + (8.9-9.0)^2 + (9.1-9.0)^2}{5-1}}$$

$$= 0.12\frac{m}{s^2}$$

This is the best estimate of the standard deviation of the parent distribution, which is the same as the error in one measurement.

So we got the best estimate of the error in one measurement.

$$\sigma_x = \sqrt{\frac{(9.0-9.0)^2+(8.8-9.0)^2+(9.1-9.0)^2+(8.9-9.0)^2+(9.1-9.0)^2}{5-1}}$$

$$= 0.12\frac{m}{s^2}$$

Therefore , the error IN THE mean OF THE 5 MEASURMENTS is

We use this formula

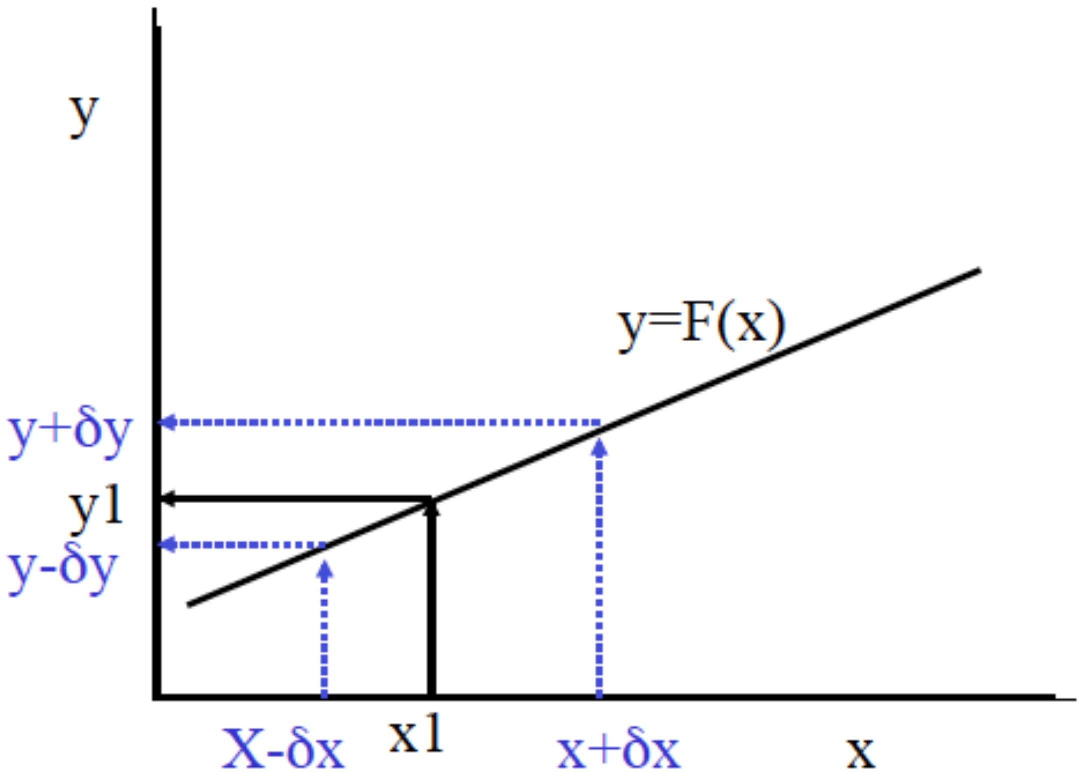
$$\sigma_m = \frac{\sigma_x}{\sqrt{N}}$$

$$\sigma_m = \frac{0.12}{\sqrt{5}} = 0.054\frac{m}{s^2}$$

$$g = 9.0\frac{m}{s^2} \pm 0.054\frac{m}{s^2}$$

ERROR PROPAGATION

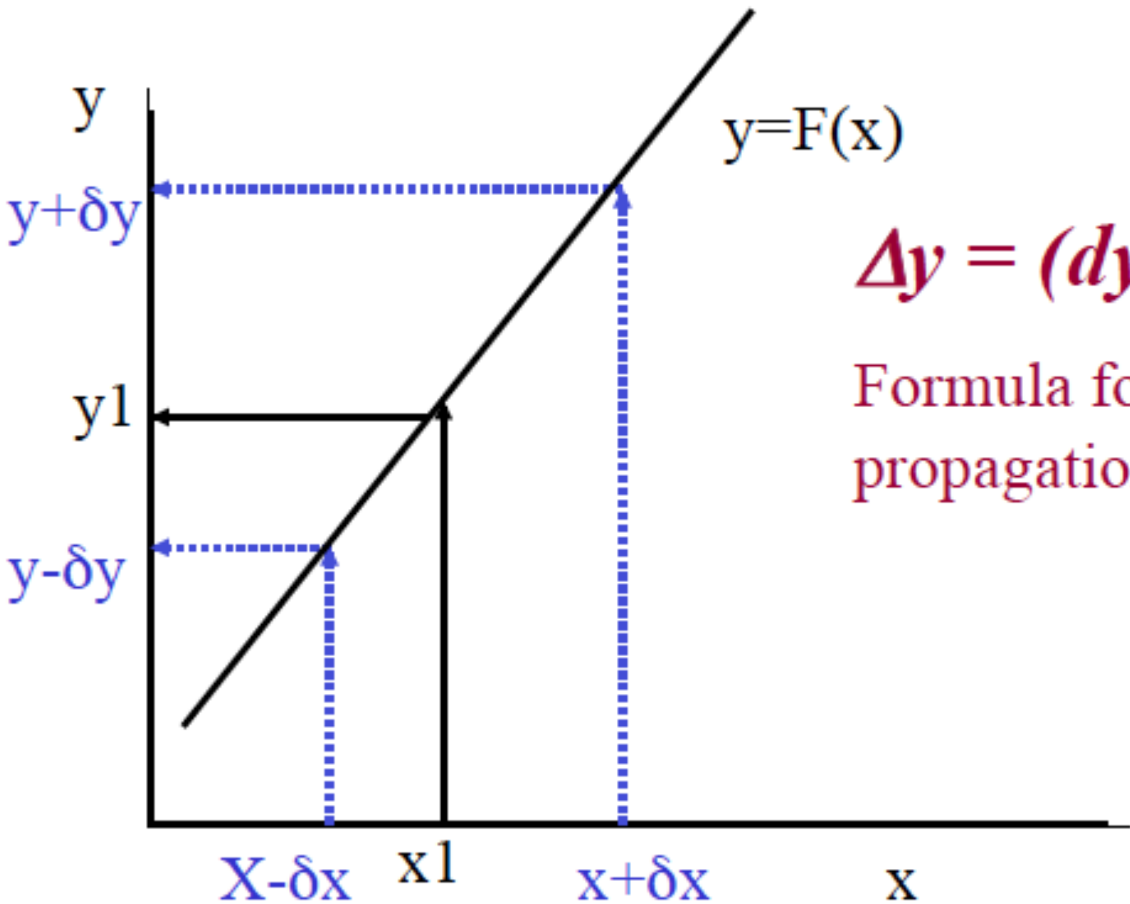
We know the error in x ((Δx)) we want to know the error in y=F(x) or (Δy)



We could plot it and project on to the y axis. But it is easier to use derivatives

ERROR PROPAGATION

We know the error in x (Δx) we want to know the error in $y=F(x)$ or (Δy)



$$\Delta y = (dy/dx)\ \Delta x$$

Formula for the propagation of errors

Propagation of Errors

The complication. What is we have more than one variable

$$x = x_o + \mathrm{v}_o t + \frac{1}{2}at^2$$

$$F = Ma$$

$$P = M\mathrm{v}$$

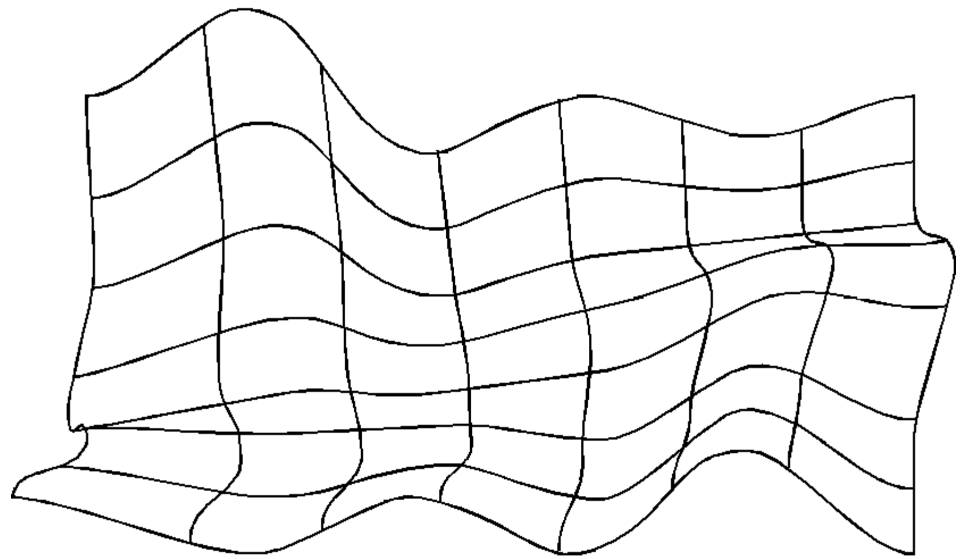
Most physical relationships involve multiple measurable variables!

y = F(x1,x2,x3,…)

Must take into account the dependence of the final measurable on each of the contributing quantities.

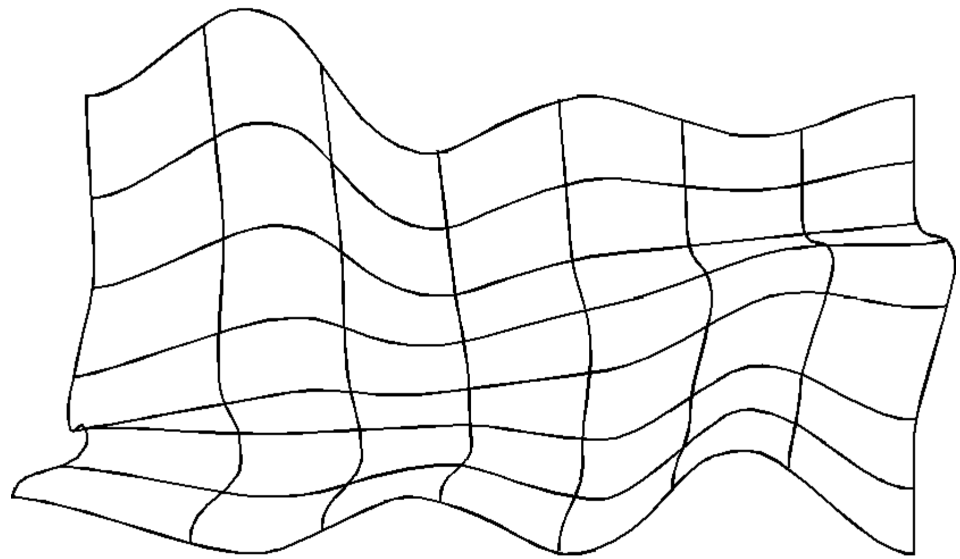
Partial derivatives

What's the slope of this graph??



For multivariable functions, one needs to define a "derivative" at each point for each variable that projects out the local slope of the graph in the direction of that variable … this is the "partial derivative".

Partial derivatives



The partial derivative with respect to a certain variable is the ordinary derivative of the function with respect to that variable where ***all the other variables are treated as constants***.

$$\frac{\partial F(x,y,z,\ldots)}{\partial x} = \left.\frac{dF(x,y,z\ldots)}{dx}\right]_{y,z\ldots const}$$

Example

$$F(x, y, z) = x^2 y z^3$$

$$\frac{\partial F}{\partial x} = 2xyz^3$$

$$\frac{\partial F}{\partial y} = x^2 z^3$$

$$\frac{\partial F}{\partial z} = x^2 y 3z^2$$

The formula for error propagation - For uncorrelated errors, errors from one source can sometimes cancel the error from another source.

If f=F(x,y,z…) and you want σ_f and you have σ_x, σ_y, σ_z …, then use the following formula:

$$\sigma_f = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 + ..}$$

Note: for correlated errors - such as a mis-calibration of an instrument or a ruler, the errors add linearly since the never cancel.

$$\sigma_f = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 + \ldots}$$

Measure of error in x

$$\sigma_f = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 + \ldots}$$

Measure of dependence of F on x

The formula for error propagation- ONLY works for uncorrelated errors.

$$\sigma_f = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 + \ldots}$$

Similar terms for each variable, add in quadrature.

For uncorrelated errors with "add in quadrature"

REPEAT: If f=F(x,y,z…) and you want σ_f and you have σ_x, σ_y, σ_z …, then use the following formula:

$$\sigma_f = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 + .}$$

Using derivatives on can show that the above equation leads to the rule that for "**string of combinations of addition or subtraction of variables**" , the **ABSOLUTE errors add in quadrature**. The string can be of any length.

Combination of errors

	Relation between Z and(A,B)	Relation between errors ΔZ and $(\Delta A, \Delta B)$
1	$Z = A + B$ $Z = A - B$	$(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$

Using derivatives you can show that the above equation leads to the rule that for " **a string of combination of multiplications and divisions of variables**" , the **FRACTIONAL errors add in quadrature. The string can be of any length**

2	$Z = AB$ $Z = A/B$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$

Combination of errors

	Relation between Z and(A,B)	Relation between errors ΔZ and $(\Delta A,\ \Delta B)$
1	$Z = A + B$ $Z = A - B$	$(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$
2	$Z = AB$ $Z = A/B$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$
3	$Z = A^n$	$\frac{\Delta Z}{Z} = n\frac{\Delta A}{A}$
4	$Z = \ln A$	$\Delta Z = \frac{\Delta A}{A}$
5	$Z = e^A$	$\frac{\Delta Z}{Z} = \Delta A$
6	$Z = \frac{A+B}{2}$	$\Delta Z = \frac{1}{2}\sqrt{\Delta A^2 + \Delta B^2}$

Table 2

From Appendix to Lab manual - discussed in workshop 1

Summary of useful equations

For uncorrelated errors

$$\sigma_f = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial F}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial F}{\partial z}\right)^2 \sigma_z^2 + \ldots}$$

$Z = AB$ $Z = A/B$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$

$$"\mu" = \bar{x} = \frac{x_1 + x_2 + \cdots + x_{N-1} + x_N}{N} = \frac{\sum_{i=1}^{N} x_i}{N}$$

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1}}$$

$$\sigma_m = \frac{\sigma_x}{\sqrt{N}}$$