Lagrangian Field Theory

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1 Introduction

This paper is a summary of Chapter 2 of Mandl and Shaw's *Quantum Field Theory* [1]. The first thing to do is to fix the notation. For the most part, we will use the same notation as Mandl and Shaw. The components of a contravariant four-vector x are denoted by x^μ for $\mu = 0, 1, 2, 3$ where $x^0 = ct$ is the time component and x^j for $j = 1, 2, 3$ are the three spatial components. Unless otherwise specified, any time a Greek letter index is used, it will range over $0, 1, 2, 3$ and any time a Latin letter index is used, it will range over $1, 2, 3$. We will also use the bold face $\mathbf{x}$ to denote the three spatial coordinates of the four-vector x. The covariant metric tensor we use is given by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.1}$$

The notation $g_{\mu\nu}$ refers to the entry in the μth row and νth column. A covariant vector is defined from the contravariant vector by the usual index-lowering

$$x_\mu = g_{\mu\nu} x^\nu \tag{1.2}$$

where the usual Einstein summation conventions are used. The contravariant metric tensor is defined by the equation

$$g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda_\nu = \begin{cases} 1 & \nu = \lambda \\ 0 & \nu \neq \lambda \end{cases} \tag{1.3}$$

so it follows that $g^{\mu\nu} = g_{\mu\nu}$ for every μ, ν, i.e. the contravariant and covariant metric tensors are the same. The metric tensor is used to define the generalized scalar product of two four-vectors. For two four-vectors a and b, their scalar product is defined as

$$ab := a^\mu b_\mu = a^\mu g_{\mu\nu} b^\nu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - (a^1 b^1 + a^2 b^2 + a^3 b^3). \tag{1.4}$$

A Lorentz transformation is a matrix Λ that preserves the scalar product xx for any four-vector x. This means that

$$g_{\mu\nu} (\Lambda x)^\mu (\Lambda x)^\nu = g_{\mu\nu} \Lambda^\mu_\alpha x^\alpha \Lambda^\nu_\beta x^\beta = g_{\mu\nu} x^\mu x^\nu. \tag{1.5}$$

We also insist that each entry of the Lorentz transformation is real. Lorentz transformations also preserve the scalar product ab for any four-vectors a, b. We now define some operators that we will need later.

$$\partial_\mu := \frac{\partial}{\partial x^\mu} \tag{1.6}$$

$$\partial^\mu := \frac{\partial}{\partial x_\mu} \tag{1.7}$$

$$\Box := \partial^\mu \partial_\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{1.8}$$

We also adopt the notation that if an index is preceded by a comma, it means we are considering the derivative with respect to that index. So for example, $F_{,\mu}$ means $\dfrac{\partial F}{\partial x^\mu}$.

2 Classical picture

We start with N fields ϕ_r for $1 \leq r \leq N$. We consider each field to be a scalar field on four-dimensional spacetime. We assume that the system can be described by a Lagrangian density

$$\mathscr{L}(\phi_r, \phi_{r,\alpha}), \tag{2.1}$$

i.e. the Lagrangian density is only a function of the fields and their first derivatives with respect to time and space. For an arbitrary region Ω in spacetime, the action integral is defined by

$$S(\Omega) \ := \ \int_\Omega d^4x \mathscr{L}(\phi_r, \phi_{r,\alpha}). \tag{2.2}$$

By imposing the usual principle of least action and insisting that $\delta S(\Omega) = 0$, we can derive the usual Euler-Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}}\right) \ = \ 0 \tag{2.3}$$

for $1 \leq r \leq N$ and $0 \leq \alpha \leq 3$.

We want to introduce the notion of a *conjugate field* (analogous to the conjugate momentum to a generalized coordinate in classic Lagrangian mechanics), but the problem is that system we are working with has uncountably many degrees of freedom. To get around this, we approximate it by a system with only countably many degrees of freedom in the following way. For a fixed time t, decompose the three-dimensional space into small cells indexed by i, each of volume $\delta\mathbf{x}_i$. If $\mathbf{x}_i$ denotes the center point of the ith cell, we approximate each field ϕ_r by letting ϕ_r take the value $\phi_r(\mathbf{x}_i)$ everywhere in the ith cell. What this accomplishes is that now we have a countable set of generalized coordinates

$$q_{ri}(t) \ := \ \phi_r(t, \mathbf{x}_i) \ =: \ \phi_r(t, i) \tag{2.4}$$

that describe the system. We can also approximate the spatial derivatives of $\phi_r(t,i)$ in terms of the values of ϕ_r in the adjacent cells. Thus the Lagrangian density for the ith cell takes the form

$$\mathscr{L}_i(\phi_r(t,i), \dot{\phi}_r(t,i), \phi_r(t,i')) \tag{2.5}$$

where the dot represents the time derivative and i' denotes the index of any cell adjacent to the ith. Then the total Lagrangian for the system is given by

$$L(t) \ = \ \sum_i \delta\mathbf{x}_i \mathscr{L}_i(\phi_r(t,i), \dot{\phi}_r(t,i), \phi_r(t,i')) \tag{2.6}$$

Now that we have discrete variables describing the system, it is possible to define the conjugate momenta in the usual way. We define

$$p_{ri} \ := \ \frac{\partial L}{\partial \dot{q}_{ri}} \ = \ \frac{\partial L}{\partial \dot{\phi}_r(t,i)} \ = \ \delta\mathbf{x}_i \pi_r(t,i) \tag{2.7}$$

where $\pi_r(t,i)$ is defined to be

$$\pi_r(t,i) \ := \ \frac{\partial \mathscr{L}_i}{\partial \dot{\phi}_r(t,i)}. \tag{2.8}$$

We now can define a Hamiltonian "density" and the usual Hamiltonian

$$\mathscr{H}_i \ := \ \pi_r(t,i)\dot{\phi}_r(t,i) - \mathscr{L}_i \tag{2.9}$$

$$H \ = \ \sum_i p_{ri}\dot{q}_{ri} - L \ = \ \sum_i \delta\mathbf{x}_i \left(\pi_r(t,i)\dot{\phi}_r(t,i) - \mathscr{L}_i\right) \ = \ \sum_i \delta\mathbf{x}_i \mathscr{H}_i. \tag{2.10}$$

At this point, we want to bring our approximation closer to the actual system by taking a limit as $\delta\mathbf{x}_i \to 0$. This gives us the following definitions and relations.

$$\pi_r(x) := \frac{\partial \mathscr{L}}{\partial \dot{\phi}_r} \quad \text{(field conjugate to } \phi_r\text{)} \tag{2.11}$$

$$L = \int_{\mathbb{R}^3} d^3\mathbf{x} \mathscr{L}(\phi_r(x), \phi_{r,\alpha}(x)) \quad \text{(Lagrangian)} \tag{2.12}$$

$$\mathscr{H}(x) := \pi_r(x)\dot{\phi}_r(x) - \mathscr{L}(\phi_r(x), \phi_{r,\alpha}(x)) \quad \text{(Hamiltonian density)} \tag{2.13}$$

$$H = \int_{\mathbb{R}^3} d^3\mathbf{x} \mathscr{H}(x) \quad \text{(Hamiltonian).} \tag{2.14}$$

Note that in analogy to classical mechanics, if the Lagrangian density does not depend explicitly on time, then the Hamiltonian is constant in time.

3 Quantum picture

Recall the discrete approximations to the system that we started with. We now want to quantize the model by interpreting the generalized coordinates and conjugate momenta as operators and imposing commutation relations on them. The commutation relations are chosen in analogy to the usual quantum-mechanical commutation relations.

$$[\phi_r(t,i), \pi_s(t,j)] := i\hbar \frac{\delta_{rs}\delta_{ij}}{\delta\mathbf{x}_i} \tag{3.1}$$

$$[\phi_r(t,i), \phi_s(t,j)] := [\pi_r(t,i), \pi_s(t,j)] := 0. \tag{3.2}$$

Now again we take a limit as $\delta\mathbf{x}_i \to 0$ and we get

$$[\phi_r(t,\mathbf{x}), \pi_s(t,\mathbf{x}')] := i\hbar\delta_{rs}\delta(\mathbf{x}-\mathbf{x}') \tag{3.3}$$

$$[\phi_r(t,\mathbf{x}), \phi_s(t,\mathbf{x}')] := [\pi_r(t,\mathbf{x}), \pi_s(t,\mathbf{x}')] := 0. \tag{3.4}$$

4 Example

This section will be dedicated to working out an example of the above theory for a specific system. Consider a system with one real-valued field ϕ and Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\phi_{,\alpha}\phi_{,}^{\alpha} - \mu^2\phi^2\right) \tag{4.1}$$

where μ is a constant. It turns out that this Lagrangian density corresponds to a spinless neutral boson with mass $\hbar\mu/c$. Using the equation of motion (2.3), we have

$$\frac{\partial \mathscr{L}}{\partial \phi} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathscr{L}}{\partial \phi_{,\alpha}}\right) \tag{4.2}$$

$$-\mu^2\phi = \frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}\phi_{,^\alpha}\right) \tag{4.3}$$

$$= \frac{1}{2}\partial_\alpha\partial^\alpha\phi \tag{4.4}$$

$$= \frac{1}{2}\Box\phi \tag{4.5}$$

so the equation of motion is

$$\left(\frac{1}{2}\Box + \mu^2\right)\phi = 0. \tag{4.6}$$

This is the Klein-Gordon equation. The field conjugate to ϕ defined by (2.11) is

$$\pi(x) \;=\; \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \;=\; \frac{1}{c^2}\dot{\phi}(x). \tag{4.7}$$

The Hamiltonian density is

$$\mathscr{H} \;=\; \pi(x)\dot{\phi}(x) - \mathscr{L} \tag{4.8}$$

$$\;=\; \frac{1}{c^2}\dot{\phi}(x)^2 - \frac{1}{2}\left(\phi_{,\alpha}\phi^{\alpha}_{,} - \mu^2\phi^2\right) \tag{4.9}$$

$$\;=\; \frac{1}{2}\left(c^2\pi(x)^2 + (\nabla\phi)^2 + \mu^2\phi^2\right). \tag{4.10}$$

The commutation relations become

$$[\phi(t,\mathbf{x}), \dot{\phi}(t,\mathbf{x}')] \;=\; i\hbar\delta(\mathbf{x}-\mathbf{x}') \tag{4.11}$$

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] \;=\; 0 \tag{4.12}$$

$$[\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] \;=\; \frac{1}{c^4}[\dot{\phi}(t,\mathbf{x}), \dot{\phi}(t,\mathbf{x}')] \;=\; 0. \tag{4.13}$$

5 Conservation laws

It is a general principle of physics that any mathematical symmetries in the Lagrangian of the system correspond to some conserved quantity in the physical system. For example, in classical mechanics, a translation-invariant Lagrangian corresponds to the conservation of energy and a rotation-invariant Lagrangian corresponds to the conservation of angular momentum. We can also apply this idea to the quantum case. Consider a transformation of a field ϕ of the form

$$\phi(x) \;\to\; \phi(x) + \delta\phi(x). \tag{5.1}$$

This will cause the Lagrangian density to change like

$$\delta\mathscr{L} \;=\; \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial\mathscr{L}}{\partial\phi_{,\alpha}}\delta\phi_{,\alpha} \;=\; \partial_\alpha\left(\frac{\partial\mathscr{L}}{\partial\phi_{,\alpha}}\delta\phi\right). \tag{5.2}$$

If the original transformation is a symmetry, then we will have $\delta\mathscr{L} = 0$, so

$$\partial_\alpha f^\alpha \;=\; 0 \tag{5.3}$$

where f^α is defined as

$$f^\alpha \;:=\; \frac{\partial\mathscr{L}}{\partial\phi_{,\alpha}}\delta\phi. \tag{5.4}$$

Now we want to investigate which quantity will be conserved as a result of this symmetry. Define

$$F^\alpha(t) \;:=\; \int_{\mathbb{R}^3} d^3\mathbf{x} f^\alpha(t,\mathbf{x}). \tag{5.5}$$

From equation (5.3), we have

$$\frac{1}{c}\frac{dF^0(t)}{dt} \;=\; -\int_{\mathbb{R}^3} d^3\mathbf{x}\partial_j f^j(t,\mathbf{x}) \;=\; 0. \tag{5.6}$$

It follows that the quantity

$$F^0 \;=\; \int_{\mathbb{R}^3} d^3\mathbf{x} f^0(t,\mathbf{x}) \tag{5.7}$$

$$\;=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\frac{\partial\mathscr{L}}{\partial\dot{\phi}}\delta\phi \tag{5.8}$$

$$\;=\; \int_{\mathbb{R}^3} d^3\mathbf{x}\pi(t,\mathbf{x})\delta\phi \tag{5.9}$$

is conserved.

6 Example

In this section we will consider an example of the theory developed in the previous section. If ϕ is a complex-valued field, then we treat ϕ and $\overline{\phi}$ as independent fields, where $\overline{\phi}$ denotes the complex conjugate of ϕ. We will suppose that the Lagrangian density $\mathscr{L}$ is invariant under infinitesimal rotations of the form

$$\phi \rightarrow \exp(i\epsilon)\phi \approx (1+i\epsilon)\phi \tag{6.1}$$

$$\overline{\phi} \rightarrow \exp(-i\epsilon)\overline{\phi} \approx (1-i\epsilon)\overline{\phi} \tag{6.2}$$

so that in the notation of the above section, we have

$$\delta\phi = i\epsilon\phi \tag{6.3}$$

$$\delta\overline{\phi} = -i\epsilon\overline{\phi}. \tag{6.4}$$

Now the conserved quantity from equation (5.9) becomes

$$F^0 = i\epsilon c \int_{\mathbb{R}^3} d^3\mathbf{x} \left(\pi(x)\phi(x) - \overline{\pi}(x)\overline{\phi}(x)\right). \tag{6.5}$$

We can scale by any constant factor we want, so we rename

$$Q := \frac{-iq}{\hbar} \int_{\mathbb{R}^3} d^3\mathbf{x} \left(\pi(x)\phi(x) - \overline{\pi}(x)\overline{\phi}(x)\right) \tag{6.6}$$

where q is an undetermined constant at the moment. We want to see how the operator Q acts on our original field ϕ, so we compute the commutator

$$[Q, \phi(x)] = \frac{-iq}{\hbar} \int_{\mathbb{R}^3} d^3\mathbf{x}' [\pi(x'), \phi(x)]\phi(x) \tag{6.7}$$

$$= -q\phi(x). \tag{6.8}$$

This result indicates that when the operator Q acts on the field ϕ, it scales it by a factor of $-q$. Similarly, if it were to act of $\overline{\phi}$, it would scale it by a factor of q. Because of this, the operators ϕ and $\overline{\phi}$ can be interpreted as creation and absorption operators for electric charge.

References

[1] F. Mandl and G. Shaw. *Quantum Field Theory.* Wiley, 1984.