The CHSH game as a Bell test thought experiment

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1 Introduction

The CHSH inequality, named after John Clauser, Michael Horne, Abner Shimony, and Richard Holt, provides an experimental framework for supporting Bell's theorem, which states that local hidden variable theories cannot explain every quantum mechanical phenomenon, particularly entanglement [1, 2]. The inequality is derived under the assumption that there exist hidden local variables and prescribes a constrain on the expected values of a Bell test experiment. Experimental violation of the CHSH inequality is therefore taken as evidence that there do not exist local hidden variables.

In this paper, we provide a description of the CHSH game, a hypothetical Bell test experiment. The players of this game can use either classical strategies (corresponding to local hidden variable theories) or quantum strategies, which involve measurements of a shared entangled bit. It can be shown that quantum strategies allow a greater winning probability than classical strategies. In fact, these maximum probabilities correspond exactly to the CHSH inequality. Hence win frequencies under a quantum strategy appear to violate the maximal winning probability prescribed by classical formalisms.

2 Background on entanglement

Suppose we have two people, Alice and Bob, who possess qubits. Consider the bipartite state

$$|0\rangle_A |0\rangle_B , \tag{1}$$

where a subscript A denotes possession by Alice and a subscript B denotes possession by Bob. It is clear that Alice's qubit is in the state $|0\rangle_A$. Similarly, Bob's qubit is assuredly in the state $|0\rangle_B$.

Now consider the state

$$|\Phi^+\rangle_{AB} \equiv \frac{\sqrt{2}}{2}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B). \tag{2}$$

It is unclear what the inidividual states of Alice's and Bob's qubits are here. This is because we cannot write $|\Phi^+\rangle_{AB}$ as a product state of the form $|\phi\rangle_A |\psi\rangle_B$. If we could, then we could say that Alice's qubit is in state $|\phi\rangle_A$ and Bob's qubit is in state $|\psi\rangle_B$. This notion of entanglement can be defined more precisely with the following definition.

Definition 2.1 (Pure-State Entanglement)**.** If a pure bipartite state cannot be written as a product state $|\phi\rangle_A |\psi\rangle_B$ for any states $|\phi\rangle_A$ and $|\psi\rangle_B$, then it is entangled.

If Alice and Bob share $|\Phi^+\rangle_{AB}$ as defined above, we say they share a bit of entanglement, or an ebit. Fig. 1 provides a diagram describing the entangled state of $|\Phi^+\rangle_{AB}$. In particular, this diagram illustrates the spatial separation between Alice and Bob and the single source that they share.

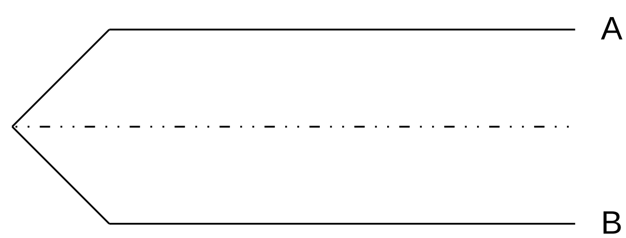

Figure 1: A diagram describing an entangled state. Note that A and B are spacially separated, but arise from a single source.

A resource is an information theoretical term that gives a metric for information processing capacity. Denote the resource of a shared ebit as

$$[qq]. \tag{3}$$

The brackets denote a noiseless resource, the letter q denotes quantum resources, and the number of letters denote the number of parties that share the resource; in this case, two.

Classical resources can also be shared. Define a bit of shared randomness as the following probability density function:

$$p_{X_A,X_B}(x_A, x_B) = \frac{1}{2}\delta(x_A, x_B). \tag{4}$$

Hence if Alice has X_A and Bob has X_B, they have both zero or one with probability 1/2. The resource of one bit of shared randomness is denoted

$$[cc], \tag{5}$$

where the letter c denotes a classical resource.

Now suppose that Alice and Bob share an ebit. Say that Alice measures the Z_A operator. (Really, Alice measures the $Z_A \bigotimes I_B$ operator, since she can't measure Bob's qubit.) That is, Alice will find $|0\rangle_A |0\rangle_B$ or $|1\rangle_A |1\rangle_B$ with probability 1/2. Hence this protocol generates a bit of shared randomness. However, note that the result of Bob measuring the Z_B operator on his qubit is already determined after Alice's measurement. That is, if Alice measures $|0\rangle_A |0\rangle_B$, she knows that Bob is guaranteed to measure the same thing. Similarly, Bob knows the results of Alice's measurement after his own. However, by Bell's inequalities, it is not possible for a bit of shared randomness to generate an entangled bit. We denote this fact by

$$[qq] \geq [cc]. \tag{6}$$

The "$\geq$" symbol indicates that there exists a protocol, such as the one described above, which generates the resource on the right by consuming the resource on the left using only local operations. The fact that shared randomness cannot be consumed to generate an entangled bit is called resource inequality.

The protocol used by Alice and Bob above generates a bit of shared randomness. There exist other protocols which can be used on an entangled bit. These protocols can generate resources other than shared randomness. In fact, shared randomness is a somewhat uninteresting resource. We will not discuss other protocols here.

3 Rules of the CHSH game

Let Alice and Bob be spatially separated such that they cannot communicate. Suppose a referee sends a uniformly chosen bit x to Alice and a bit y to Bob. Alice and Bob then send bits a and b, respectively, back to the referee. If

$$x \wedge y = a \oplus b, \tag{7}$$

where $\wedge$ denotes a logical AND operation and $\oplus$ denotes a logical XOR operation, then Alice and Bob win the game. Fig. 2 illustrates the gameplay.

Let the function V be defined as

$$V(x, y, a, b) = \begin{cases} 1 & \text{if } x \wedge y = a \oplus b, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

This function V acts as an indicator for Alice's and Bob's win frequency as a function of their received and sent bits. Since the bits x and y are defined as being chosen uniformly, the probability that any given pair of such bits is chosen is given by

$$p_{XY}(x, y) = \frac{1}{4}, \tag{9}$$

for any x, y. The probability that Alice and Bob return a given pair of bits a, b is given by

$$p_{AB|XY}(a, b|x, y), \tag{10}$$

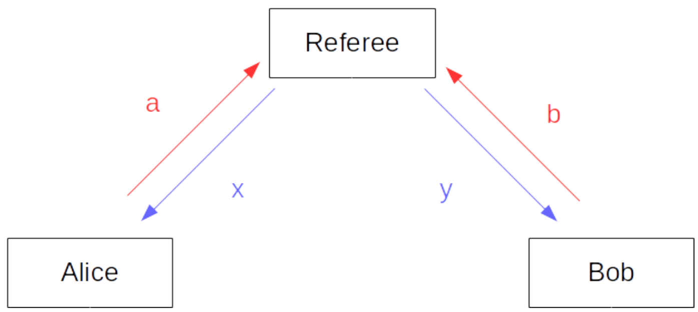

Figure 2: A diagram of the CHSH game rules. Arrows in blue represent step one, where the referee sends bits to Alice and Bob. Arrows in red represent step two, where Alice and Bob send bits back.

which indicates that their choice of returned bits is dependent on the bits that they receive from the referee. From these definitions, we can write that the probability that Alice and Bob win the game is given by

$$\begin{aligned}\Pr\{\text{win}\} &= \sum_{a,b,x,y} V(x,y,a,b) p_{AB|XY}(a,b|x,y) p_{XY}(x,y) \\ &= \frac{1}{4} \sum_{a,b,x,y} V(x,y,a,b) p_{AB|XY}(a,b|x,y). \end{aligned} \tag{11}$$

Let us consider the conditional probability density function $p_{AB|XY}$ more closely. As stated above, Alice and Bob are spatially separated and cannot communicate. However, they are allowed to confer before the game and discuss a strategy for choosing bits to send back to the referee based on the bits they receive. The probability density function $p_{AB|XY}$ therefore depends upon the strategy used by the players.

We want a way to mathematically describe their strategies. Let the random variable Λ encode their strategy. Then we can rewrite $p_{AB|XY}$ as

$$p_{AB|XY}(a,b|x,y) = \int \mathrm{d}\lambda\, p_{AB|\Lambda XY}(a,b|\lambda,x,y) p_{\Lambda|XY}(\lambda|x,y), \tag{12}$$

where $\lambda \in \Lambda$. This formulation will allow us to quantify the winning probabilities of generalized strategies. In particular, we can contrast the winning probabilities of classical strategies with those of quantum strategies. We shall show that, if Alice and Bob use a classical deterministic strategy, their maximal winning probability is $3/4$; if they use a quantum strategy, their maximal winning probability is $1/2 + \sqrt{2}/4$. This suggests that some quantum mechanical effects cannot be described in any possible classical theory.

4 Maximal winning probability with classical strategies

Using a classical strategy, the random variable Λ corresponds to correlations that Alice and Bob can share before the game. Since the bits x and y are chosen independently at random, Λ cannot depend on x nor y. Hence

$$p_{\Lambda|XY}(\lambda|x,y) = p_{\Lambda}(\lambda). \tag{13}$$

Recall that Alice and Bob are separated and therefore acting independently. Thus we can write

$$p_{AB|\Lambda XY}(a,b|\lambda,x,y) = p_{A|\Lambda XY}(a|\lambda,x,y)p_{B|\Lambda XY}(b|\lambda,x,y). \tag{14}$$

Since Alice does not have access to bit y and Bob does not have access to bit x, this further simplifies to

$$p_{AB|\Lambda XY}(a,b|\lambda,x,y) = p_{A|\Lambda X}(a|\lambda,x)p_{B|\Lambda Y}(b|\lambda,y). \tag{15}$$

We can therefore rewrite Eq. 12 as

$$\begin{aligned} p_{AB|XY}(a,b|x,y) &= \int \mathrm{d}\lambda\, p_{AB|\Lambda XY}(a,b|\lambda,x,y)p_{\Lambda|XY}(\lambda|x,y) \\ &= \int \mathrm{d}\lambda\, p_{A|\Lambda X}(a|\lambda,x)p_{B|\Lambda Y}(b|\lambda,y)p_{\Lambda}(\lambda). \end{aligned} \tag{16}$$

Now note that we can always find a random variable N such that

$$p_{A|\Lambda X}(a|\lambda,x) = \int \mathrm{d}n\, f(a|\lambda,x,n)p_N(n), \tag{17}$$

where $n \in N$, for some deterministic binary-valued function f. Similarly, there exists a random variable M such that

$$p_{B|\Lambda Y}(b|\lambda,y) = \int \mathrm{d}m\, g(b|\lambda,y,m)p_M(m), \tag{18}$$

where $m \in M$, for some deterministic binary-valued function g. Hence we have

$$p_{AB|XY}(a,b|x,y) = \iiint \mathrm{d}\lambda\, \mathrm{d}n\, \mathrm{d}m\, f(a|\lambda,x,n)g(b|\lambda,y,m)p_{\Lambda}(\lambda)p_N(n)p_M(n). \tag{19}$$

We can force the random variable Λ to encode M and N. As a result, we find

$$p_{AB|XY}(a,b|x,y) = \int \mathrm{d}\lambda\, f'(a|\lambda,x)g'(b|\lambda,y)p_{\Lambda}(\lambda), \tag{20}$$

where f' and g' are again deterministic and binary-valued. We can now write Eq. 11 as

$$\begin{aligned}\Pr\{\text{win}\} &= \frac{1}{4}\sum_{a,b,x,y} V(x,y,a,b)p_{AB|XY}(a,b|x,y) \\ &= \int \mathrm{d}\lambda\, p_\Lambda(\lambda)\left[\frac{1}{4}\sum_{x,y,a,b} V(x,y,a,b)f'(a|\lambda,x)g'(b|\lambda,y)\right] \qquad (21) \\ &\le \frac{1}{4}\sum_{x,y,a,b} V(x,y,a,b)f'(a|\lambda^*,x)g'(b|\lambda^*,y), \qquad (22)\end{aligned}$$

where $\lambda^* \in \Lambda$ is chosen to be such that the inequality holds. We can do this because the quantity in Eq. 21 represents an average over Λ, and so there must exist $\lambda \in \Lambda$ such that the integrand is greater than the integral. This fact suggests that the winning probability is maximized using a deterministic strategy.

A deterministic strategy entails that Alice chooses a bit a_x dependent on the received bit x and Bob chooses a bit b_y dependent on his received bit y. Table 1 lists the possible choices for x and y and the result of $a_x \oplus b_y$ for that choice. For each choice of x and y, Alice and Bob win if the equation in the corresponding row is satisfied.

x	y	$x \wedge y$	$= a_x \oplus b_y$
0	0	0	$= a_0 \oplus b_0$
0	1	0	$= a_0 \oplus b_1$
1	0	0	$= a_1 \oplus b_0$
1	1	1	$= a_1 \oplus b_1$

Table 1: All possible outcomes of a CHSH game where Alice's and Bob's strategies are classical and deterministic. Each row corresponds to a possible choice of bits x and y.

Since addition modulo 1 is commutative and, for any bit $c \in \{0,1\}$, $c \oplus c = 0$, the fourth column of Table 1 adds to zero modulo one. The third column, however, adds to one modulo one. Therefore, there does not exist a choice of a_x and b_y such that all four equations are satisfied.

The next best conceivable strategy is one that satisfies three of the four equations. Indeed, such a classical strategy exists. Suppose that Alice and Bob both send back 0, regardless of what bits x and y they receive from the referee. It can be clearly seen that this strategy results in a winning probability of 3/4. Hence a classical strategy affords a maximal winning probability of 3/4.

Let the quantity α represent the winning probability minus the losing probability for some classical strategy. Since the two probabilities must add to one and the winning probability is bounded above by 3/4, α is bounded above by 1/2. Now let $S = 4\alpha$. Then we have that

$$\|S\| \le 2, \qquad (23)$$

which is exactly the CHSH inequality. The classical strategies we have discussed here correspond to local hidden variable theories; the random variable Λ encoded general local hidden variables.

5 Maximal winning probability with quantum strategies

With a quantum strategy, we allow Alice and Bob to share an entangled quantum state $|\phi\rangle_{AB}$, although Alice and Bob themselves are still separated and cannot otherwise communicate. Now Λ can correspond to strategies based off of measurements on this state $|\phi\rangle_{AB}$. A quantum strategy will involve Alice and Bob performing some measurement on their qubit based on the bit they receive from the referee. Denote Alice's x-dependent local projective measurements as

$$\{\Pi_a^{(x)}\}, \tag{24}$$

where $\sum_a \Pi_a^{(x)} = I$. Similarly, denote Bob's y-dependent local projective measurements as

$$\{\Pi_b^{(y)}\}, \tag{25}$$

where $\sum_b \Pi_b^{(y)} = I$. Then we can rewrite Eq. 12 as

$$p_{AB|XY}(a,b|x,y) = \langle\phi|_{AB}\Pi_a^{(x)} \otimes \Pi_b^{(y)}|\phi\rangle_{AB}\,. \tag{26}$$

Hence, for a given quantum strategy (choice of measurements), we can rewrite the winning probability from Eq. 11 as

$$\Pr\{\text{win}\} = \frac{1}{4}\sum_{a,b,x,y} V(x,y,a,b)\,\langle\phi|_{AB}\Pi_a^{(x)} \otimes \Pi_b^{(y)}|\phi\rangle_{AB}\,. \tag{27}$$

Suppose the input bits x, y are 01, 10, or 00. Then Alice and Bob win if they both send back the same bit. The probability that Alice and Bob do this is given by

$$\langle\phi|_{AB}\Pi_0^{(x)} \otimes \Pi_0^{(y)}|\phi\rangle_{AB} + \langle\phi|_{AB}\Pi_1^{(x)} \otimes \Pi_1^{(y)}|\phi\rangle_{AB}\,, \tag{28}$$

whereas the probability that they send back different bits is given by

$$\langle\phi|_{AB}\Pi_0^{(x)} \otimes \Pi_1^{(y)}|\phi\rangle_{AB} + \langle\phi|_{AB}\Pi_0^{(x)} \otimes \Pi_1^{(y)}|\phi\rangle_{AB}\,. \tag{29}$$

Let

$$A^{(x)} \equiv \Pi_0^{(x)} - \Pi_1^{(x)}, \tag{30}$$

$$B^{(y)} \equiv \Pi_0^{(y)} - \Pi_1^{(y)}. \tag{31}$$

Then the probability of winning minus that of losing can be written as

$$\langle\phi|_{AB}A^{(x)} \otimes B^{(y)}|\phi\rangle_{AB} \tag{32}$$

if it is not the case that both x and y are 1.

Now suppose conversely that both x and y are 1. From a similar calculation to that above, the probability of winning minus that of losing in this case is

$$-\langle\phi|_{AB} A^{(x)} \otimes B^{(y)} |\phi\rangle_{AB}. \tag{33}$$

Let

$$C_{AB} \equiv A^{(0)} \otimes B^{(0)} + A^{(0)} \otimes B^{(1)} + A^{(1)} \otimes B^{(0)} - A^{(1)} \otimes B^{(1)}. \tag{34}$$

Then in general the probability of winning minus that of losing is given by

$$\frac{1}{4} \langle\phi|_{AB} C_{AB} |\phi\rangle_{AB}. \tag{35}$$

Note that, with some algebraic manipulation,

$$C_{AB}^2 = 4I_{AB} - \left[A^{(0)}, A^{(1)}\right] \otimes \left[B^{(0)}, B^{(1)}\right]. \tag{36}$$

Using the properties of operator norms, we find that the infinity norm of C_{AB}^2 is bounded above:

$$\|C_{AB}^2\|_\infty \leq 8. \tag{37}$$

This implies that

$$\|C_{AB}\|_\infty \leq 2\sqrt{2}, \tag{38}$$

which is the upper bound for the CHSH inequality under a quantum mechanical formalism. This constraint on C_{AB} is exactly analagous to that imposed on the quantity S in Eq. 23. Hence it seems that some observable phenomena are not explainable with local hidden variables.

We have shown that, with a quantum strategy, the winning probability minus the losing probability is bounded above by $2\sqrt{2}$. Since these probabilities add to 1, the winning probability is bounded above by $1/2 + \sqrt{2}/4$. The inequality in Eq. 38 is not strict, as there does indeed exist a strategy with this maximal winning probability; however, we will not describe it here.

6 Conclusions

In this paper, we described how the CHSH inequality implied that quantum mechanical phenomena could not be entirely described in classical terms with local hidden variable theories. The CHSH game described here constitutes an example of a Bell test. Within the CHSH game, we have shown that classical strategies afford a maximum winning probability of $3/4$, whereas quantum strategies expand this maximum winning probability to $1/2 + \sqrt{2}/4$. These two strategies correspond to the CHSH inequality derived using local hidden variable theories and quantum mechanical mathematics, respectively. Hence the CHSH game represents a profound result in modern physics: that quantum mechanical effects are inherently indescribable using classical mechanical formalisms.

Notes

This paper represents a reproduction of my lecture notes on entanglement, which is from Section 3.6 of Wilde's book on Quantum Information Theory [3]. I have cut down much of the material and added some useful references to get a better picture of the utility and historical context of the content. Consider the text of this paper to be a sort of script and the numbered equations and figures to be the things that need to be put on the chalkboard.

References

[1] J. S. Bell, *Speakable and unspeakable in quantum mechanics: Collected papers on quantum philosophy*, Cambridge university press, 2004.

[2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Proposed experiment to test local hidden-variable theories*, Physical review letters, 23 (1969), p. 880.

[3] M. M. Wilde, *Quantum information theory*, Cambridge University Press, 2013.