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Angular Momentum and the Clebsh-Gordan Decomposition

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1 INTRODUCTION

The following is meant as a brief overview of specific topics relating to angular momentum in the context of quantum mechanics, inclusing the Clebsh-Gordan decomposition and the Wigner-Eckhart theorem. Most of what follows here is based off chapter IV.3 of Anthony Zee's book *Group Theory in a Nutshell for Physicists*.

2 QUANTIZATION OF ANGULAR MOMENTUM

In 1913, in order the solve the outstanding issues with the planetary atomic model, Niels Bohr introducted the assumption that the angular momentum L is quantized by multiples of $\hbar$. Bohr introduced the assumption that the lowest angular momentum state for the hydrogen atom has $L = mvr = \hbar$, which allowed for a definite value to be assigned to the Bohr radius and the minimum energy state of the hydrogen atom. Recall that $SO(3)$ is a Lie algebra with a set of generators $\vec{L}_{\text{Lie}}$. The angular momentum operators $\vec{L}$ are given by $\hbar\vec{L}_{\text{Lie}}$. This means that the operators satisfy the following relation:

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \tag{2.1}$$

where ϵ_{ijk} is the Levi-Civita symbol. If we leave L constant and take $\hbar$ down to zero, we expect that this should reduce to the behavior of a classical system. Indeed, as we see, the commutators of the $\vec{L}$ operators drop to 0, they approach the behavior of a classically commuting variable. By extension, as $\vec{L}^2|l,m\rangle = \hbar l(l+1)|l,m\rangle$ and $L_z|l,m\rangle = \hbar mketl,m$, it follows that if L remains fixed and $\hbar \to 0$, $l, m \to \infty$. This reproduces what should be a familiar quantum

mechanical fact: that quantum systems approach classical ones as its quantum numbers go to infinity.

3 Addition of Angular Momentum

Consider a system consisting of two particles in a spherically symmetric potential. Call these particles “prime" and “un-prime", with states $|l,m\rangle$ and $\left|l',m'\right\rangle$, respectively. If the two particles have no interaction, then the Hamiltonian's eigenstates will simply be a direct product of the individual particles states, i.e. $|l,m\rangle \otimes \left|l',m'\right\rangle$. If the particles *do* interact on the other hand, there will be a reaction term H_I dependent on the distance between the particles.

Suppose for example, that $l = 2$, $l' = 1$. Then the un-prime wavefunction will be a product of a purely radial function and a function of ϕ and θ which transforms like the traceless and symmetric tensor S^{ij}. S^{ij} has five components, each of which is a function of ϕ and θ. These correspond to the five possible values for m, $-2,-1,0,1$, and 2. Similarly, the wavefunction for the prime particle will be a product of a radial function and an angular function which transforms like a vector T^k. This vector has three componenets, which correspond to the three possble values for m: -1, 0, and 1. Recall that in general we have

$$l \otimes l' = (l+l') \oplus (l+l'-1) \oplus ... \oplus (|l-l'|+1) \oplus (|l-l'|) \tag{3.1}$$

(equation 4.2.11 in Lee's book). So in this particular case this relation is:

$$2 \otimes 1 = 3 \oplus 2 \oplus 1. \tag{3.2}$$

So our original 15 states are in fact three classes of degenerate states, with degeneracies 7, 5, and 3.

4 Multiplication of Ladders

By multiplying tensors together, we can obtain larger irreducible representations, which is essentially what we did above in Eq. 3.1. We do something similar with Lie algebras, multiplying kets to obtain larger representations. Let j and j' be irreduciple representations for $SO(3)$. We then have two collections of kets:

$$\{\left|j,m\right\rangle \mid m \in \{-j,-j+1...j-1,j\}\}$$

and

$$\{\left|j',m'\right\rangle \mid m' \in \{-j',-j'+1...j'-1,j'\}\}.$$

When acted on by the generators J_i, the prime kets will transform to linear combinations of themselves, and similarly the unprime ones will transform to linear combinations of the

unprime kets.

The question is now what are the product kets $\left|j,m\right\rangle \otimes \left|j',m'\right\rangle$? We know there are $(2j+1)(2j'+1)$ such states, which the generators J_i send to linear combinations, producing a $(2j+1)(2j'+1)$-dimensional representation of $SO(3)$, which will be reducible in the sense that their representing matrices will be block diagonalizable. This approach is effectively equivalent to the approach using tensors. There is a bijective correspondence between the kets $\left|j,m\right\rangle$ and the components of the tensor $T^{i_1 \cdots i_j}$, so the act of multiplying tensors and multiplying kets are in this context effectively equivalent.
Consider an infinitessimal rotation around the z-axis given by $R \simeq I + i\theta J_z$. Clearly both $\left|j,m\right\rangle$ and $\left|j',m'\right\rangle$ will also rotate in reaction to this. Observe:

$$\begin{aligned}
\left|j,m\right\rangle \otimes \left|j',m'\right\rangle &\to R\left|j,m\right\rangle \otimes R\left|j',m'\right\rangle \\
&\simeq (I + i\theta J_z)\left|j,m\right\rangle \otimes (I + i\theta J_z)\left|j',m'\right\rangle \\
&= (I + i\theta m)\left|j,m\right\rangle \otimes (I + i\theta m)\left|j',m'\right\rangle \\
&\simeq (1 + i\theta(m+m'))\left(\left|j,m\right\rangle \otimes \left|j',m'\right\rangle\right) + O(\theta^2)
\end{aligned}$$

Equivalently, this means that

$$\begin{aligned}
J_z\left(\left|j,m\right\rangle \otimes \left|j',m'\right\rangle\right) &= \left(\left(J_z\left|j,m\right\rangle\right) \otimes \left|j',m'\right\rangle\right) + \left(\left|j,m\right\rangle \otimes \left(J_z\left|j',m'\right\rangle\right)\right) \\
&= (m+m')\left(\left|j,m\right\rangle \otimes \left|j',m'\right\rangle\right)
\end{aligned}$$

Therefore we can see J_z acts on $\left|j,m\right\rangle$ and $\left|j',m'\right\rangle$ separately via a sort of "product rule" form, and that th eigenvalues of the two states simply add directly.

(To reduce notation, $\left|j,m\right\rangle \otimes \left|j',m'\right\rangle$ will be written as $\left|j,j',m,m'\right\rangle$ from here on)

5 The Clebsh-Gordan Decomposition

First, an example. Consider $j = \frac{1}{2}$, $j' = \frac{1}{2}$. There will be $(2j+1)(2j'+1) = 4$ states, corresponding to $m, m' = \pm\frac{1}{2}$ (from here on we will write $\left|j,j',m,m'\right\rangle = \left|m,m'\right\rangle$, as j and j' are fixed for the time being.) Our four states are as follows:

$$\begin{aligned}
&\left|\frac{1}{2},\frac{1}{2}\right\rangle \\
&\left|-\frac{1}{2},\frac{1}{2}\right\rangle \\
&\left|\frac{1}{2},-\frac{1}{2}\right\rangle \\
&\left|-\frac{1}{2},-\frac{1}{2}\right\rangle
\end{aligned}$$

The representation produced by these states can be decomposed into a smaller number of irreducible states indexed by a number J. These irreducible representations will be written as

$|J,M\rangle$ for $M=-J...J$. Our basic approach will be to take the highest eigenvalue state $\left|\frac{1}{2},\frac{1}{2}\right\rangle$ and repeatedly apply the J_- operator. This state has eigenvalue $\frac{1}{2}+\frac{1}{2}=1$, so we have

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = |1,1\rangle \tag{5.1}$$

Where the ket on the left is $\left|m,m'\right\rangle$ and the ket on the right is $|J,M\rangle$. So we apply J_- on both sides. First, we recall the following relations:

$$J_-\left|\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle,\ J_-|1\rangle = \sqrt{2}|0\rangle,\ J_-|0\rangle = \sqrt{2}|-1\rangle$$

Using these, and the action of J_z on product kets found in the previous section, we conclude that

$$J_-\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

Therefore

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) \tag{5.2}$$

Via another application of the J_- operator we get $|1,-1\rangle = \left|-\frac{1}{2},-\frac{1}{2}\right\rangle$. The last remaining state is $\frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)$. This state we can see has eigenvalue 0, so it corresponds to $|0,0\rangle$. To summarize:

$$\begin{aligned}
|J,M\rangle &= \left|m,m'\right\rangle \\
|1,1\rangle &= \left|\frac{1}{2},\frac{1}{2}\right\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) \\
|0,1\rangle &= \frac{1}{\sqrt{2}}\left(\left|-\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right) \\
|1,-1\rangle &= \left|-\frac{1}{2},-\frac{1}{2}\right\rangle
\end{aligned}$$

In other words,

$$\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$$

This process that we have just worked through is the Clebsh-Gordan decomposition. The coefficients, such as $\frac{1}{\sqrt{2}}$, are called the Clebsh-Gordan coefficients. The decomposition is found from:

$$|J,M\rangle = \sum_{m=-j}^{j}\sum_{m'=-j'}^{j'}\left|j,j',m,m'\right\rangle\left\langle j,j',m,m'\middle|J,M\right\rangle \tag{5.3}$$

Which we can view as being essentially a rearrangement of the relation

$$\sum_{m=-j}^{j}\sum_{m'=-j'}^{j'} \left|j,j',m,m'\right\rangle\left\langle j,j',m,m'\right| = I,$$

which should be familiar.

6 Wigner-Eckhart Theorem

Early atomic spectroscopy faced a number of challenged that puzzled physicists of the time. Notably, many transition lines which were expected to appear did not, and the reason why they did not appear was not yet understood. Suppose we have a perturbation, which may cause a transition from some initial state $|i\rangle$ to a final state $\left|f\right\rangle$. The probability of this transition occurring will in general be expressed as $\left\langle f\right|\mathcal{O}\left|i\right\rangle$, where $\mathcal{O}$ is some operator. In the context of atomic spectroscopy, $|i\rangle$ and $\left|f\right\rangle$ will transform like members of an irreducible representation for $SO(3)$, where we have $|i\rangle = \left|\alpha,j,m\right\rangle$ and $\left|f\right\rangle = \left|\alpha',j',m'\right\rangle$. Here α and α' denote quantum numbers which are not directly relevant to the structure of $SO(3)$, such as principal quantum number and so on. $\mathcal{O}$ will also transform like an element of an irreducible representation. Denote by $\mathcal{O}_{JM}$ the operator corresponding to the state $|J,M\rangle$.
The Wigner-Eckhart theorem states that

$$\begin{aligned}\left\langle \alpha',j',m'\right|\mathcal{O}_{JM}\left|\alpha,j,m\right\rangle &= \left(\left\langle j',m'\right|\left(|JM\rangle\otimes\left|j,m\right\rangle\right)\right)\left\langle \alpha',j'\right||O_J|\left|\alpha,j\right\rangle \\ &= \left\langle j',m'\right|\left.J,j,M,m\right\rangle\left\langle \alpha',j'\right||O_J|\left|\alpha,j\right\rangle\end{aligned}$$

Which means we can reduce the initial probability amplitude into two factors, $\left\langle j',m'\right|\left.J,j,M,m\right\rangle$ and $\left\langle \alpha',j'\right||O_J|\left|\alpha,j\right\rangle$. The latter of these two is called the reduced matrix element of $\mathcal{O}$. Evaluation of this term must be done on the basis of physics, beginning from the SchrÃűdinger equation and computing integrals directly. This quantity will depend on α,α',m,m',J,j, and j', but not on m or m'.
Note that the other term, $\left\langle j',m'\right|\left.J,j,M,m\right\rangle$ is a Clebsch-Gordan coefficient. This is logical, as $\mathcal{O}_{JM}\left|\alpha,j,m\right\rangle$ transforms in the same way as $|J,M\rangle\otimes\left|j,m\right\rangle = \left|J,j,M,m\right\rangle$. Note also that $\left\langle j',m\right|\left.J,j,M,m\right\rangle = 0$ unless we have $j'\in\{|j-J|,|j-J|+1,\ldots j+J-1,j+J\}$ and $m' = M+m$. Alternatively, we can rewrite these conditions in terms of $\Delta j = j'-j$ and $\Delta m = m'-m$:

$$\begin{aligned}|\Delta j| = |j'-j| &\leq J \\ \Delta m = m'-m = M &\leq J\end{aligned}$$

The intensity of an observed emission line, i.e. the probability of the emission occurring, is given by the absolute square of $\left\langle \alpha',j',m'\right|\mathcal{O}_{JM}\left|\alpha,j,m\right\rangle$, which means the intensity of the various lines is determined by group relations. In addition, the conditions above also put strict limits on which lines that it is possible to observe. This gives rise to the now well-known selection rules which puzzled early 20^{th} century physicists. For (α,j), (α',j') a pair of initial and final states, respectively, we expect, for a given transition type fixed by J, a total of

$(2j+1)(2j'+1)$ possible transitions. Many of these do not occur because of the selection rules listed above, while the frequency under which the remaining ones occur is fixed entirely by group theory, as the reduced matrix element $\langle \alpha', j' \big| |O_J| \big| \alpha, j \rangle$ cancels out completely.