# From Gravitational Collapse to Black Holes

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## 1 Introduction

The purpose of this independent study is to familiarize ourselves with the fundamental concepts of general relativity and, in particular, black-hole physics. In this paper, I examine the spacetime geometry of a star collapsing into a black-hole.

Towards the end of its lifetime, a star has burnt up almost all of its fuel. As the thermonuclear fusion process begins to slow down and release less energy, the star begins to contract due to the force of gravity. For stars of about the mass of the Sun, the gravitational collapse is usually halted by degeneracy pressure by the Pauli exclusion principle (two fermions cannot be at the same quantum state). However, if the mass of the remnant exceeds $3 - 4$ solar masses (the initial mass of the star exceeds 25 solar masses), in order for degeneracy pressure to support against gravitational collapse, the electrons have to move faster than the speed of light. As a result, the star collapses into what is known as a *black hole*. The gravitational influence of a black hole is so strong that even light. cannot escape it. To understand the spacetime geometry of a black hole, it is crucial that we understand the geometry of a collapsing star.

This paper is based on my lecture for the Kapitza Society. In Section 2, I will introduce the Schwarzschild geometry and use it to describe the geometry around a spherically symmetrical black hole. In Section 3, I will examine a spherically symmetrical collapse of a star. I will also generate the spacetime diagram using *Python* to describe the Schwarzschild geometry.

## 2 The Schwarzschild Black Hole

### 2.1 The Schwarzschild Geometry

In 1916, Karl Schwarzschild solved Einstein's field equations and derived the spacetime geometry outside of a spherically symmetrical body. The Schwarzschild geometry could be used to describe the geometry outside of a collapsing star if we make the approximation that the collapse is spherically symmetric. Note that this is hardly the case for any collapsing body because the rotation of the body around its axis will distort the shape. However, if the rotation angular velocity is small enough, the body is spherically symmetrical to the first order approximation.

The line element in Schwarzschild geometry outside of a body of mass $M$ is given by:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

Here $t$ is the time measured by a stationary observer at infinity, *theta* and *phi* are the polar and azimuthal angle, and $r$ is the radius (measured as the circumference, divided by $2\pi$, of a sphere centered around the body). To simplify our equation, we use the natural unit where the gravitational constant $G$ and the speed of light $c$ are equal to 1 ($G = c = 1$). The corresponding metric is:

$$g_{\alpha\beta} = \begin{bmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{2}$$

The Schwarzschild geometry breaks down at $r = 0$ and $r = 2M$, the latter called the *Schwarzschild radius*. These are known as *singularities*. The singularity at $r = 2M$ is a coordinate singularity (it can be removed by changing the coordinate system), while the singularity at $r = 0$ is a physical singularity (it cannot be removed by changing the coordinate system and indicates some special properties of spacetime).

## 2.2 Eddington-Finkelstein Coordinates

When a star collapses, its radius approaches the Schwarzschild radius. In such cases, the Schwarzschild coordinate system faces many difficulties in describing the spacetime geometry due to the singularity at $r = 2M$ and the change in the signs of $g_{tt}$ and $g_{rr}$ in Eq. 2. However, because $r = 2M$ is a coordinate singular, we may be able to overcome it by defining a different parameterization for the Schwarzschild geometry.

We define a parameter $v$ such that:

$$t = v - r - 2M\log\left|\frac{r}{2M} - 1\right| \tag{3}$$

Expressing $dt^2$ in terms of $dv$ and $dr$, the line element in Eq. 1 becomes:

$$dt^2 = dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dvdr \tag{4}$$

$$\Rightarrow \boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)} \tag{5}$$

This is known as the Eddington-Finkelstein coordinates. The Eddington-Finkelstein coordinates are especially useful for studying gravitational collapse because the singularity at

$r = 2M$ vanishes (i.e. it connects smoothly the spacetime regions inside and outside this radius). On the contrary, the singular at $r = 0$ does not vanish, indicating that there is some special properties of spacetime at this radius. For the study of gravitational collapse, we only consider the case where $r > 0$.

The corresponding metric of the Eddington-Finkelstein coordinate $(v, r, \theta, \phi)$ is:

$$g_{\alpha\beta} = \begin{bmatrix} -(1-2M/r) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix} \tag{6}$$

Unlike the Schwarzschild metric, the non-diagonal terms $g_{vr}$ and $g_{rv}$ do not vanish. On the other hand, like the Schwarzschild metric, at large $r$, the metric is approximately flat and $t \approx v - r$ for the logarithm term in Eq. 3 vanishes.

## 2.3 Light cones in Eddington-Finkelstein coordinates

In general relativity, to understand a spacetime geometry, we have to examine the behavior of a light ray in said geometry. In this section, we solve for the equation of motion of a light ray described by the Eddington-Finkelstein coordinates.

Consider a radial light ray. By definition, the world line of the light ray will have $d\theta = d\phi = 0$ (radial) and $d^2 = 0$ (null interval). The line element in Eq. 5 gives the differential equation;

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr = 0 \tag{7}$$

The first and most obvious solution is $dv = 0$, or the light travels along the curve:

$$\boxed{v = \text{const}} \quad \text{or} \quad \boxed{t + r + 2M \log\left|\frac{r}{2M} - 1\right| = \text{const}} \tag{8}$$

For $r < 2M$ and $r > 2M$, $r$ decreases as $t$ increases. For an observer as infinity, this is thus an equation of an ingoing radial light ray.

The second solution can be obtained by solving the reduced equation:

$$-\left(1 - \frac{2M}{r}\right) dv + 2dr = 0 \tag{9}$$

$$\Rightarrow \int dv = \int -2\left(1 - \frac{2M}{r}\right)^{-1} dr \tag{10}$$

$$\Rightarrow \boxed{v - 2r - 4M \log\left|\frac{r}{2M} - 1\right| = \text{const}} \quad \text{or} \quad \boxed{t - r - 2M \log\left|\frac{r}{2M} - 1\right| = \text{const}} \tag{11}$$

When $r > 2M$, the solution corresponds to an outgoing light ray, because $r$ increases as $t$ increases. On the other hand, when $r < 2M$, the solution corresponds to an ingoing light ray, because $r$ decreases as $t$ increases. As a result, we see that light behaves differently inside and outside of the radius $r = 2M$. When $r = 2M$, the $dv$ term in Eq. 9 vanishes. We obtain the third solution to the differential equation $dr = 0$, or $r = 2M$.

By solving for the equation of motion of a radial light ray, we find the regions inside and outside of the radius $r = 2M$ have different physical properties. To further examine these differences, we examine the spacetime diagram. We plot $\tilde{t} = v - r$ versus $r$. As briefly mentioned above, $\tilde{t} = v - r$ is the time measured by an observer infinitely far from the body. In Fig. 1, each dotted blue line corresponds to a possible solution of $v =$ const, while each blue line corresponds to a possible solution of Eq. 11. The black vertical line corresponds to the solution $r = 2M$, and the bold vertical line at $r = 0$ corresponds to the physical singularity. At each point in the spacetime diagram, a future light cone is defined by a

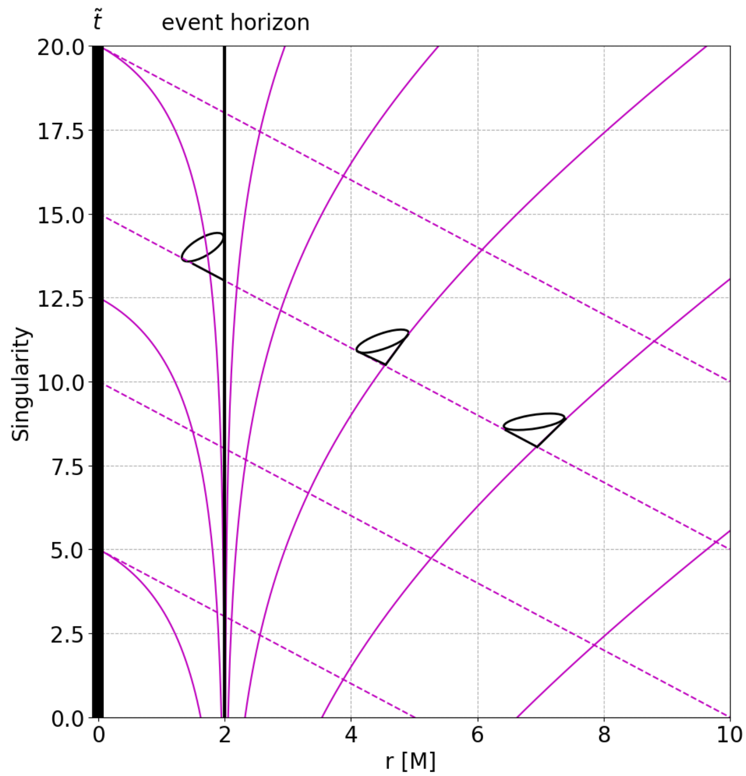


Figure 1: The spacetime diagram of the Schwarzschild geometry in the Eddington-Finkelstein coordinates. As the radius decreases, the future light cone gets increasingly tipped towards the singularity at $r = 0$. At $r \leq 2M$, all timelike geodesics will head towards the singularity. At $r = 2M$, a null geodesic allows for light to orbit the black hole in a circular motion.

solution of $v =$ const and a solution of Eq. 11. In Fig. 1, three light cones are shown at the intersections. As the radius $r$ decreases, the light cone is increasingly tipped towards the singularity at $r = 0$. At the boundary $r = 2M$, one radial light ray points towards the singularity, while another orbits the black hole in a circular orbit. At $r \leq 2M$, all future light cones will point towards the singularity.

## 2.4 The Event Horizon

Because at each point in spacetime, no object with mass may move outside of the future light cone (as they would be moving faster than the speed of light), any object falling pass

$r = 2M$ has to move *faster* than the speed of light to avoid being sucked into the singularity! This implies no information inside $r = 2M$ can be obtained, and thus regions inside and outside this boundary are physically disconnected! The boundary $r = 2M$ is called the *event horizon* of the black hole because it divides spacetime into two regions with different physical properties. It is generated by light rays that neither escape to infinity or fall into the singularity. Outside the event horizon, light rays may escape into infinity. Inside the event horizon, light rays can no longer escape and will always fall into the singularity. Hence, physicists name these objects "black holes". It is (almost) impossible to detect a black hole via light-based telescope[1]; its location can only be inferred from its massive gravitational influence on nearby stars or the emission gravitational waves.

Consider in the case of $r = \text{const}$ $(dr = 0)$, the line element becomes:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{12}$$

$$= \begin{cases} +\left|1 - \frac{2M}{r}\right| dv^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) > 0, \text{ for } r < 2M \\ -\left|1 - \frac{2M}{r}\right| dv^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) > 0, \text{ for } r > 2M \end{cases} \tag{13}$$

When $r < 2M$, the term $g_{vv}$ in Eq. 6 is positive, so the line element is *spacelike*. In contrast, when $r > 2M$, the line element is *timelike*. This is also evident from Fig. 1. The vertical line $r = \text{const}$ is always inside the local light cone at $r > 2M$. However, at $r < 2M$, the local light cone is tipped such that the vertical line is always lying outside of the cone. At the event horizon, the vertical line contributes to one of the two null curves. Outside $r = 2M$, $r = \text{const}$ is a space in spacetime, while inside $r = 2M$, $r = \text{const}$ is a time in spacetime. Geometrically, once we cross the event horizon, space and time trades place with each other!

At any time $t$ $(dt = 0)$, the surface of the event horizon of a Schwarzschild black hole is described by

$$d\Sigma^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) = (2M)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{14}$$

The *area of the event horizon* thus given by

$$\boxed{A_{\text{surface}} = 16\pi M^2} \tag{15}$$

This area is time-independent and can only be changed by altering the mass of the black hole. Because nothing can escape the black hole, the mass, and by extension, the area of the event horizon, cannot classically decrease[1].

# 3 Gravitational Collapse

## 3.1 The Collapsing Dust Sphere

When a star collapses due to its own gravity, the materials inside the surface of the star get compressed and heated up. As the density and temperature increases, radiation pressure,

[1] A black hole can still be detected by Hawking radiation. Theoretically predicted by Stephen Hawking in 1974, it is the radiation due to the capture of a spontaneous particle-antiparticle pair just outside of the event horizon. Because a particle or antiparticle escapes via Hawking radiation, it is responsible for the loss in mass and energy of a black hole.

which is proportional to the fourth power of temperature, goes up dramatically and pushes the materials outward, counteracting gravitational collapse. Furthermore, in some cases, the increase in density, temperature, and pressure may also re-ignite nuclear fusion and prolong the life of the star. For the purpose of our study, we will ignore these effects and consider the gravitational collapse of a sphere of non-interacting, pressureless dust. Because the Schwarzschild geometry describes only spacetime outside a spherically symmetric body, we consider only what happens outside of the dust sphere.

The equation of motion for a radially plunge orbit is:

$$r(\tau) = \left(\frac{3}{2}\right)^{\frac{2}{3}} (2M)^{\frac{1}{3}} (\tau_\star - \tau)^{\frac{2}{3}} \tag{16}$$

where $\tau$ is the proper time, or the time measured by an observer riding along with the surface of the dust sphere, and $\tau_\star$ is an integration constant at $r = 0$. For a distant observer, the radius of the sphere is instead given by the equation:

$$t = t_\star + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} - 2\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 2\log\left[\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 1\right] - \log\left|\left(\frac{r}{2M}\right)^{\frac{1}{2}} - 1\right|\right] \tag{17}$$

Substitute in Eq. 3, we arrive at the equation of motion for the surface of a spherically symmetrical collapsing dust sphere in the coordinate $(v, r, \theta, \phi)$:

$$v - r = t_\star + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} - 2\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 2\log\left[\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 1\right]\right] \tag{18}$$

$$\Rightarrow \boxed{\frac{v(r)}{2M} = \frac{r}{2M} - \frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} - 2\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 2\log\left[\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 1\right] + \text{const}} \tag{19}$$

Once the surface of a star collapses to a radius smaller than the radius of the event horizon $r = 2M$, the radius of the star can only decrease. The material on the surface must move along a geodesic inside the future local light cone, as indicated by special relativity, and all light cones are point towards the singularity. Hence, the star will inevitably collapse into an infinitely dense singularity at $r = 0$.

## 3.2 Gravitational Redshift

Consider a distant observer at $r_R$ and an observer riding along with the surface of a collapsing star. The falling observer communicates with the distant observer by sending signals at a frequency of $\omega_\star$. Before the falling observer crosses the event horizon, the signal can still reach the distant observer. The signal is emitted at a time interval $\Delta\tau = 2\pi/\omega_\star$. Let the signal emitted at $(v_E, r_E)$ be received by the distant observer at the proper time $t_R$. The time interval between successive signals is $\Delta t_R(t_R)$ is a function of $t_R$.

The equation of motion of the signal is described by Eq. 11:

$$v - 2r - 4M\log\left|\frac{r}{2M} - 1\right| = \text{const}$$

Because this holds true for all $(v, r)$, the constant on the right-hand sided at $(v_R, r_R)$ is equal to the constant at $(v_E, r_E)$. For $r_E \approx 2M$, the logarithm term dominates and thus the left-handed side becomes:

$$v_E - 2r_E - 4M \log \left| \frac{r_E}{2M} - 1 \right| \approx -4M \log \left| \frac{r_E}{2M} - 1 \right| \tag{20}$$

On the other hand, for $r_R \gg 2M$, the logarithm term vanishes. Furthermore, $t_R \approx v_R - r_R$. The left-handed side becomes:

$$v_R - 2r_R - 4M \log \left| \frac{r_R}{2M} - 1 \right| \approx v_R - 2r_R \approx t_R - r_R \tag{21}$$

We find the relation between $r_R$, $r_E$, and $t_R$ to be

$$-4M \log \left| \frac{r_E}{2M} - 1 \right| \approx t_R - r_R \tag{22}$$

$$\Rightarrow r_E = 2M \left[ 1 + \exp\left( \frac{r_R - t_R}{4M} \right) \right] \tag{23}$$

Take the derivative with respect to $t_R$, and assuming that the interval $\Delta t_R \ll 1$:

$$\frac{\Delta r_E}{\Delta t_R} \approx \frac{dr_E}{dt_R} = \frac{-1}{2} \exp\left( \frac{r_R - t_R}{4M} \right) \tag{24}$$

If the falling observer has a four-velocity of $u^\alpha = (u^v, -|u^r|, 0, 0)$ (the negative sign is due to the radius decreasing), then we may express the space interval between successive signals $\Delta r_E$ as the time interval between successive signals $\Delta \tau$: $\Delta r_E = -|u^r| \Delta \tau$. We may express $\Delta t_R$ in terms of $\Delta \tau$ and $t_R$:

$$\frac{\Delta r_E}{\Delta t_R} = \frac{-|u^r| \Delta \tau}{\Delta t_R} = \frac{-1}{2} \exp\left( \frac{r_R - t_R}{4M} \right) \tag{25}$$

$$\Rightarrow \Delta t_R = 2|u^r| \Delta \tau \exp\left( \frac{t_R - r_R}{4M} \right) \tag{26}$$

$$\Rightarrow \omega_R = \frac{\omega_\star}{2|u^r|} \exp\left( \frac{r_R - t_R}{4M} \right) \tag{27}$$

$$\Rightarrow \boxed{\omega_R \propto \omega_\star \exp\left( -\frac{t_R}{4M} \right)} \tag{28}$$

We have expressed the received frequency as a function of the emitted frequency and the proper time of the distant observer. The constant of proportionality is $\exp(r_R/4M)/2|u^r|$. As long as the radial component of the four-velocity does not change, Eq. 28 will hold. Because the exponential term is decreasing as $t_R$ increases, the distant observer will see the signal more redshifted over time. Because the photon energy is proportional to its frequency, the observer will see the signal less luminous over time. Once the falling observer crosses the event horizon $r = 2M$, the signals, along with any particle with non-zero mass, can no longer reach the distant observer but instead fall towards the singularity $r = 0$. As indicated above, there can be no information exchanged between the two observers.

To examine the complete picture, we plot the spacetime diagram $\tilde{t}$ versus $r$ in Fig. 2. The surface of the star is described in Eq. 19. The shaded region is inside the surface of the star and thus cannot be described by the Schwarzschild geometry. The dotted lines are the world lines of the signal from the falling observer. From Fig. 2, as the falling observer is heading towards the event horizon, the time interval between successive signals as observed by the distant observer gets larger and larger. Once the falling observer crosses the event horizon, no signal can escape as all null geodesics point towards the singularity.

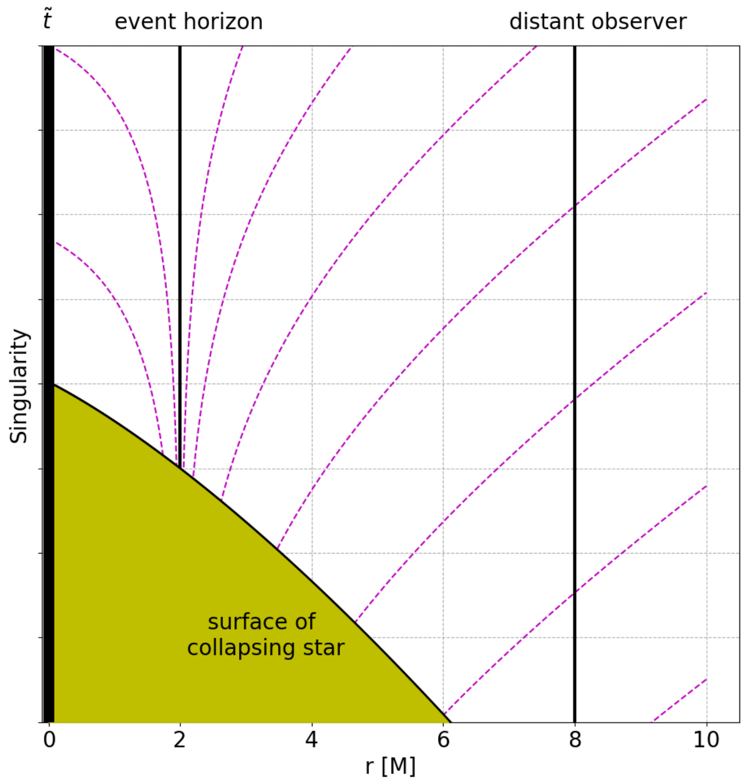


Figure 2: The surface of a collapsing star in the Eddington-Finkelstein coordinates. As the falling observer gets closer to the event horizon, the distant observer will see time interval between successive signals longer and the signals more redshifted. Once the falling observer falls inside the event horizon, no signal can escape and the communication between the two observers is interrupted.

# 4 Conclusion

In this paper, we used the Eddington-Finkelstein coordinates to describe the Schwarzschild geometry of a spherically symmetrical collapsing star. We derived the null geodesics of a radial light ray and found that inside and outside the event horizon, light ray behaves qualitatively differently. Outside the horizon, light rays may escape into infinity or fall towards the horizon. However, inside the horizon, all light rays cannot escape and will instead converge into a singularity. This singularity corresponds to an infinitely dense region of spacetime, where the laws of physics as we know break down. We also derived the equation of motion of the surface of a collapsing dust sphere and examined the case of the two observers. One observer is infinitely far from the collapsing star, while the other rides along on the surface of the star. We found as the surface of the star collapses, the signals emitted by the falling observer will get exponentially redshifted as seen by the distant observer.

All of our derivations are based on the assumption that the collapse is spherically symmetrical. This is not necessarily the case because any rotation of the body around an axis will distort the shape. However, this is our first step to understanding the geometry of a nonspherical collapse and eventually the geometry of a black hole. The following properties of a spherical collapse will hold in the case of a nonspherical collapse: the formation of a singularity, the formation of an event horizon, and the increase in the area of the horizon.

# References

[1] J.B. Hartle. *Gravity: An Introduction to Einstein's General Relativity.* Addison-Wesley, 2003.