Introduction to and Simple Applications of the Boltzmann Equation

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1 Introduction

One of the most important assumptions in cosmology is that the universe is both isotropic (meaning any direction in space should be indistinguishable from any other) and homogeneous (meaning any region of space should be the same as any other). However, it is apparent that this assumption is only true for certain scales (see figure 1). Thus, the primary motivation for the Boltzmann Equation is the need to track anisotropies and inhomogeneities in the universe.

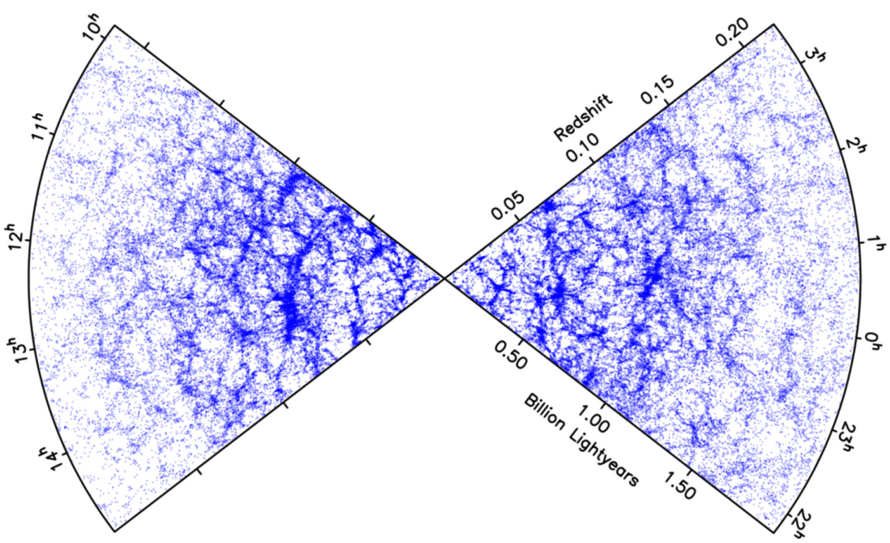

Figure 1: Visual representation of data from the 2df Galaxy Redshift Survey. Filaments and voids in the galactic foam are visible evidence of the inhomogeneity of matter distribution. However, when considering a large enough scale, it also appears that the assumption of homogeneity is valid.

2 The Boltzmann Equation

The Boltzmann equation is, put simply, the way a distribution changes with time. Thought of this way, the form presented in eq.(1) makes intuitive sense,

$$\frac{df}{dt} = C\left[f\right] \tag{1}$$

where it is important to remember that C[f] is merely an equation governing all collisions relevant to the distributions of interest. Evidently, in a system with no collisions, C[f] will equal zero since the distribution should not change in this situation. Although ultimately the goal of the Boltzmann equation is to track all the perturbations of the major matter constituents of the universe, this paper will merely consider the Boltzmann Equation as applied to the simple harmonic oscillator and collisionless photon distribution.

3 Non-relativistic Simple Harmonic Oscillator

Considering the movement of a non-relativistic particle in a simple harmonic oscillator (SHO), it is evident that the location/distribution function should likely be dependent on time, position, and momentum. Then, since position and momentum are both functions of time in an SHO, we can write the Boltzmann equation as follows,

$$\frac{df(x,p,t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} \tag{2}$$

Since there are no collisions an SHO, we can assume that eq.(2) is also equal to zero. To find the differential equation allowing us to solve for the distribution function, evidently we must find the values of dx/dt and dp/dt. Luckily, these are easily found by referring to the equations of motion. For dx/dt we merely need consider the non-relativistic definition of momentum,

$$\frac{dx}{dt} \equiv \frac{p}{m} \tag{3}$$

And for dp/dt simply recall that for an SHO,

$$\frac{dp}{dt} = -kx \tag{4}$$

Hence, eq.(1) becomes,

$$\frac{df(x,p,t)}{dt} = \frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} + -kx\frac{\partial f}{\partial p} = 0 \tag{5}$$

Looking at eq.(5) we notice a few things. First of all, the term describing the change in the distribution function with position has a coefficient equal to velocity. i.e. this term is telling us the speed of the particle in the oscillator.

Second off, the third term has a coefficient equal to the restoring force, which tells us the speed of the change of the momentum of the particle in the oscillator.

Obviously, as for any differential equation, we need some initial condition for the distribution function in order to solve the Boltzmann equation. However, even without this there is still important information that we can gain from the Boltzmann equation alone. Recall that the energy of an SHO is as so,

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{6}$$

Assuming our system is in equilibrium, the distribution function should not change over time and so $\partial f/\partial t = 0$. Since now the distribution function and energy depend only on x and p, we can write eq.(5) as,

$$\frac{df(x,p)}{dt} = \frac{df(E)}{dt} = \frac{p}{m}\frac{\partial f(E)}{\partial x} + -kx\frac{\partial f(E)}{\partial p} = \frac{df}{dE}\left[\frac{p}{m}\frac{\partial E}{\partial x} - kx\frac{\partial E}{\partial p}\right] \tag{7}$$

In general, if a system is in equilibrium, then the distribution function can be written as a function of energy and even in systems with collisions, since the distribution will not change, we can expect the collision terms of the Boltzmann equation to cancel out. As a result, the distribution function will often be of a familiar form (e.g. the Maxwell-Boltzmann distribution)

4 Collisionless Photons

In a smooth, unperturbed, and expanding universe we use the Friedmann-Robinson-Walker (FRW) metric,

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{bmatrix} \tag{8}$$

However, since the Boltzmann equation necessitates that there be perturbations in the universe, we must modify this metric. To do this we simply two functions of space and time, Ψ and Φ. Although it is possible to add more functions describing different types of perturbations, it turns out that for photons these are the most relevant and hence we can ignore these. The metric then becomes,

$$g_{\mu\nu} = \begin{bmatrix} -1-2\Psi(t,\vec{x}) & 0 & 0 & 0 \\ 0 & a^2(1+2\Phi(t,\vec{x})) & 0 & 0 \\ 0 & 0 & a^2(1+2\Phi(t,\vec{x})) & 0 \\ 0 & 0 & 0 & a^2(1+2\Phi(t,\vec{x})) \end{bmatrix} \tag{9}$$

Where Ψ describes perturbations due to Newtonian potential and Φ describes spatial curvature perturbations. Typically, Ψ and Φ are small enough such that we can ignore terms of order 2 or greater.

Assuming that we can, we want to emulate the method in section 3 of expressing the Boltzmann equation in terms of partial derivatives. To do this, we first must realize that the distribution function of the photon likely must also depend on time, position, and momentum. Considering the space-time point $x^\mu = (t, \vec{x})$, we can describe the momentum as

$$P^\mu \equiv \frac{dx^\mu}{d\lambda} \tag{10}$$

Where λ is a strictly increasing parameterization of the path of the photon. Although the distribution function is defined in 8 dimensions, it turns out that there are only 3 independent components of the photon's momentum as a result of the fact that its masslessness necessitates that eq.(11) be true:

$$P^2 \equiv g_{\mu\nu} P^\mu P^\nu = 0 \tag{11}$$

In order to find the most convenient choice for the independent components, we must first consider eq.(11) in the perturbed FRW metric. This gives us,

$$P^2 = -(1 + 2\Psi)(P^0)^2 + |\vec{p}|^2 = 0 \tag{12}$$

Then solving for P^0 yields,

$$P^0 = \frac{|\vec{p}|}{\sqrt{1 - \Psi}} \cong |\vec{p}|(1 - \Psi) \tag{13}$$

Where by convention $\Psi < 0$ implies the photon is in a dense region. The physical consequences of this choice are made apparent by considering the movement of a photon out of a dense region (i.e. a potential well). Evidently, P^0 will be smaller outside of the region verses inside of it, indicating that the energy of the photon is decreasing as it transitions between the two regions. This, in the context of light, means that the photon is being redshifted, which is what we would normally expect.

In essence, what eq.(13) allows us to do is replace P^0 with $|\vec{p}|$. This will come in handy when cleaning up the Boltzmann equation, which can now be written as,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{dx^i}{dt} + \frac{\partial f}{\partial |\vec{p}|} \frac{d|\vec{p}|}{dt} + \frac{\partial f}{\partial \hat{p}^i} \cdot \frac{d\hat{p}^i}{dt} \tag{14}$$

This equation can be simplified by considering that the Bose-Einstein distribution function, eq.(15), is equal to the zeroth order distribution function.

$$f_{BE} = \frac{1}{e^{(E-\mu)/T} - 1} \tag{15}$$

As we can see, this function does not depend on the direction of the momentum ($\hat{p}$), hence if we assume $\partial f / \partial \hat{p}^i$ is non-zero then it must be at least a first-order term. However, $d\hat{p}^i/dt$ must also be a first-order term because in the absence of any potentials, a photon will travel in a straight line. As a result of this, the product of these two terms (i.e. the last term in eq.(14)) is at least a second-order term, which means we can ignore it.

Now, if we remember eq.(10) and consider that $x^0 = t$, then we can do as follows,

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0} \tag{16}$$

Since, as we've established, it makes things easier if we do everything in terms of $|\vec{p}|$ and $\hat{p}$, we want to relate P^i to $\hat{p}^i$. $\hat{p}^i$ is just a unit vector of P^i, so obviously we can just write,

$$P^i = C\hat{p} \tag{17}$$

Where C is some constant. The magnitude of momentum then becomes,

$$|\vec{p}| = aC\sqrt{1 - 2\Phi} \tag{18}$$

Where solving for C allows us to write eq.(17) as,

$$P^i = |\vec{p}|\hat{p}^i \frac{1 - \Phi}{a} \tag{19}$$

Combining eq.(16), eq.(19), and eq.(13) then allows us to write,

$$\frac{dx^i}{dt} = \frac{\hat{p}^i}{a}(1 + \Psi - \Phi) \tag{20}$$

Where the sign convention is that in a dense region, $\Phi > 0$ and $\Psi < 0$. This does not jive with our expectations, since we'd normally predict that a photon would slow in a dense region (and dx/dt is velocity).

Using essentially the same argument as with $\partial f/\partial \hat{p}^i$, we know $\partial f/\partial x^i$ must also be at least a first-order term. As a result, we can also neglect the second-order products of $\partial f/\partial x^i$ and dx^i/dt in eq.(14). The last term we're interested in is dp/dt. For this we will have to first start with the time component of the geodesic equation, which can be written as,

$$\frac{dP^0}{d\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta \tag{21}$$

Using eq.(13) allows us then to write

$$\frac{d|\vec{p}|}{dt}(1 - \Psi) = |\vec{p}|\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{|\vec{p}|}(1 + \Psi) \tag{22}$$

Multiplying both sides by $(1 + \Psi)$ and then dropping the second-order terms gives us

$$\frac{d|\vec{p}|}{dt} = |\vec{p}|\left[\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{|\vec{p}|}(1 + 2\Psi) \tag{23}$$

Focusing on $\Gamma^0_{\alpha\beta} P^\alpha P^\beta / |\vec{p}|$ as the major unknown, we can write this as

$$\Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{|\vec{p}|} = \frac{g^{0\nu}}{2} \left[2\frac{\partial g_{\mu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right] \frac{P^\alpha P^\beta}{|\vec{p}|} \tag{24}$$

From here, it's just a lot of math to show that eq.(24) = eq.(25),

$$\Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{|\vec{p}|} = \frac{2\Psi - 1}{2} \left[2|\vec{p}|\frac{\partial \Psi}{\partial t} - 4\frac{\partial \Psi}{\partial x^\beta} P^\beta - |\vec{p}| \left(2\frac{\partial \Psi}{\partial t} + 2H[1 + 2\Phi] \right) (1 - 2\Phi) \right] \tag{25}$$

Throwing out second-order terms and setting $(1-2\Phi)$ to 1 when multiplying by $\partial\Phi/\partial t$ allows us to write

$$\Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{|\vec{p}|} = (1 - 2\Psi) \left[\frac{\partial \Psi}{\partial t} |\vec{p}| + 2\frac{\partial \Psi}{\partial x^i} \frac{|\vec{p}|\hat{p}^i}{a} + |\vec{p}| \left(\frac{\partial \Phi}{\partial t} + H \right) \right] \tag{26}$$

Feeding eq.(26) into eq.(22) and sorting out terms then gives us

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} \tag{27}$$

Each of the terms in eq.(27) has a physical meaning. For instance, $-H$ shows that, as the universe expands, our photon is losing momentum. It is left up to the reader to recall the sign conventions used for Φ and Ψ and consider potential wells in order to understand the other two terms.

We finally have enough information to write the Boltzmann equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - |\vec{p}|\frac{\partial f}{\partial p} \left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} \right] \tag{28}$$

5 Conclusion

This brings us to the end of this section covering the basics of the Boltzmann equation. Assuming that the reader is interested in continuing the sequential derivation for the Boltzmann equation in a perturbed universe, they can read "Boltzmann Equation for Photons" by Tyler Perlman as a direct continuation of this piece, and then "The Boltzmann Equation for Photons and Dark Matter" by Jeremy Atkin as a continuation of that.

6 Resources

[1] Chapters 4 and 2 of *ModernCosmology* by Scott Dodelson
[2] The 2dF Galaxy Redshift Survey (http://www.2dfgrs.net/)