Vector fields

Differential Geometry
Chapter I, O'Neill U Cook
Summary

(1)

- $V(\vec{p}) = (v_1(\vec{p}), v_2(\vec{p}), v_3(\vec{p}))_p$, where v_i are realvalued functions.

$= v_1(\vec{p})(1,0,0)_p + v_2(\vec{p})(0,1,0)_p + v_3(\vec{p})(0,0,1)_p$

$= v_1(\vec{p})U_1(\vec{p}) + v_2(\vec{p})U_2(\vec{p}) + v_3(\vec{p})U_3(\vec{p})$

Claim: If V is a vector field on $\mathbb{R}^3$, there are three uniquely determined real-valued functions v_1, v_2, v_3 on $\mathbb{R}^3$ $\ni$ $V = \sum_i v_i U_i$ where U_i are vector fields on $\mathbb{R}^3$ $\ni$ $U_i(\vec{p}) = (0, \dots, \underset{i\text{th}}{1}, \dots, 0)_p$ for each point $\vec{p}$ of $\mathbb{R}^3$. We call them the natural frame fields on $\mathbb{R}^3$.

If $V = 2(x_1U_1 + x_2U_2) - x_1(U_1 - (x_2)^2U_3)$

$= x_1U_1 + 2x_2U_2 + x_1(x_2)^2U_3$

$V(\vec{p}) = x(\vec{p})U_1(\vec{p}) + 2y(\vec{p})U_2(\vec{p}) + x(\vec{p})(y(\vec{p}))^2U_3(\vec{p})$

but by definition, $x(\vec{p}) = p_1$; $y(\vec{p}) = p_2$; $z(\vec{p}) = p_3$

so $V(\vec{p}) = p_1U_1(\vec{p}) + 2p_2U_2(\vec{p}) + p_1(p_2)^2U_3(\vec{p})$ Put $x = x_1$, $y = x_2$, $z = x_3$.

If $V = xU_1 - y^2U_3$ and $f = x^2y + z^3$, then

$V[f] = xU_1[f] - y^2U_3[f]$

$= xU_1[x^2y + z^3] - y^2U_3[x^2y + z^3]$

(if $V = (x, 0, -y^2)$ find $V(\vec{p})[f]$ for $\vec{p} = (p_1, p_2, p_3)$ and $f = x^2y + z^3$.)

Question after showing that $U_i[f] = \frac{\partial f}{\partial x_i}$

But $U_i(\vec{p})[f] = (0, 0, \dots, \underset{i\text{th}}{1}, \dots, 0)_{\vec{p}}[f]$, and $\vec{v}_p[f] = \sum_j v_j \frac{\partial f}{\partial x_j}\Big|_{\vec{p}}$

where $\vec{v}_p = (v_1, v_2, \dots, v_n)$.

$$= \sum_j u_j \frac{\partial f}{\partial x_j}\Big|_{\vec{p}} = \frac{\partial f}{\partial x_i}\Big|_{\vec{p}}$$

So $U_i[f] = \frac{\partial f}{\partial x_i}$

giving $V[f] = 2x^2y - 3y^2z^2$

and $V(\vec{p})[f] = 2(p_1)^2p_2 - 3(p_2)^2(p_3)^2$

1-Forms.

Def: A 1-form ϕ on $\mathbb{R}^3$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^3$ $\ni$ ϕ is linear at each point.

$$\phi_p : T_p(\mathbb{R}^3) \to \mathbb{R}$$

Def: If f is a differentiable real-valued function on $\mathbb{R}^3$, the differential df of f is the 1-form $\ni$

$$df(\vec{v}_p) = \vec{v}_p[f] \quad \forall \text{ tangent vectors } \vec{v}_p$$

Indeed real valued from $T_p(\mathbb{R}^3) \to \mathbb{R}$ and linear since

$$\vec{v}_p[f] = \sum_i v_i \frac{\partial f}{\partial x_i}$$

recall that $\vec{v}_p[f] = \frac{d}{dt}\left(f(\vec{p}+t\vec{v})\right)_{t=0}$

So $df(\vec{v}_p)$ knows all initial rates of change of f in all direction on $\mathbb{R}^3$, so it's not surprising that differentials are fundamental to the calculus on $\mathbb{R}^3$.

real valued function $\mathbb{R}^3 \to \mathbb{R}$

Lemma: If ϕ is a 1-form on $\mathbb{R}^3$, then $\phi = \sum f_i dx_i$, where $f_i = \phi(U_i)$.

These functions f_1, f_2, f_3 are called the Euclidean coordinate functions of ϕ.

Proof. ϕ and $\sum f_i dx_i$ are equal iff $\phi(\vec{v}_p) = (\sum_i f_i dx_i)(\vec{v}_p)$ $\forall \vec{v}_p \in T_p(\mathbb{R}^3)$

But $\left(\sum_i f_i dx_i\right)(\vec{v}_p) = \sum_i f_i(\vec{p})\, dx_i(\vec{v}_p)$

but $dx_i(\vec{v}_p) = \vec{v}_p(x_i) = \sum_j v_j \frac{\partial x_i}{\partial x_j}\Big|_{\vec{p}} = v_i$ (independent of $\vec{p}$!)

giving $\left(\sum_i f_i dx_i\right)(\vec{v}_p) = \sum_i v_i f_i(\vec{p})$

On the other hand, $\phi(\vec{v}_p) = \phi\left(\sum v_i U_i(\vec{p})\right) = \sum v_i \phi(U_i(\vec{p})) = \sum_i v_i f_i(\vec{p})$

QED

Corollary: If f is a differentiable function on $\mathbb{R}^3$, then
$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

Proof: $df(\vec{v}_p) = \vec{v}_p[f] = \sum_i v_i \frac{\partial f}{\partial x_i}\Big|_{\vec{p}}$

$$\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right)(\vec{v}_p) = \sum_i \frac{\partial f}{\partial x_i}(\vec{p})\, dx_i(\vec{v}_p) = \sum_i v_i \frac{\partial f}{\partial x_i}\Big|_{\vec{p}}$$

QED

Ex: If ϕ is a 1-form, find $\phi(V)$ for any vector field V.

$\Rightarrow$ $\phi = \sum_i f_i dx_i$ where $f_i = \phi(U_i)$

$V = \sum_j v_j U_j$

Since $\phi(V)$ is linear, i.e., $\phi(fV + gW) = f\phi(V) + g\phi(W)$

$$\phi(V) = \phi\Big(\sum_j v_j U_j\Big) = \sum_j v_j\, \phi(U_j) = \sum_j v_j f_j$$

Ex: Evaluate the 1-form $\phi = x^2 dx - y^2 dz$ on the vector field

$V = xy(U_1 - U_3) + yz(U_1 - U_2)$

$$\Rightarrow \quad V = \underbrace{(xy + yz)}_{v_1} U_1 + \underbrace{(-yz)}_{v_2} U_2 + \underbrace{(-xy)}_{v_3} U_3$$

$$\phi = \underbrace{x^2}_{f_1} dx \underbrace{- y^2}_{f_3} dz, \quad f_2 = 0$$

$$\text{so } \phi(V) = \sum_j v_j f_j = (xy + yz)x^2 + (-yz)0 + (-xy)(-y^2)$$
$$= x^3y + x^2yz + xy^3$$

Mappings

Def: Given a function $F: \mathbb{R}^n \to \mathbb{R}^m$, let $\underbrace{f_1, f_2, \ldots, f_m}_{\text{Euclidean coordinate functions of } F}$ denote the real-valued functions on $\mathbb{R}^n$ ∋

$$F(\vec{p}) = (f_1(\vec{p}), f_2(\vec{p}), \ldots, f_m(\vec{p})) \quad \forall \vec{p} \in \mathbb{R}_n$$

We write $F = (f_1, f_2, \ldots, f_m)$

- The function F is differentiable provided its coordinate functions are differentiable. A differentiable function $F: \mathbb{R}^n \to \mathbb{R}^m$ is called a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Ex: $F: \mathbb{R}^3 \to \mathbb{R}^3$, if $F = (x-y, x+y, 2z)$

then $F(\vec{p}) = (x(\vec{p}) - y(\vec{p}), x(\vec{p}) + y(\vec{p}), 2z(\vec{p}))$, where $\vec{p} = (p_1, p_2, p_3)$

$$= (p_1 - p_2, p_1 + p_2, 2p_3)$$

Def: Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\vec{v}$ is a tangent vector to $\mathbb{R}^n$ at $\vec{p}$) (basically $\vec{v} \in T_p(\mathbb{R}^n)$), let

$F_*(\vec{v})$ be the initial velocity of the curve $\beta(t) = F(\vec{p} + t\vec{v})$.

Then $F_{*p}: T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ going to see why soon.

↳ is called the tangent map of F.

Proposition: Let $F = (f_1, f_2, \ldots, f_m)$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $\vec{v} \in T_p(\mathbb{R}^n)$, then

$$F_*(\vec{v}_p) = (\vec{v}_p[f_1], \vec{v}_p[f_2], \ldots, \vec{v}_p[f_m])_{F(p)}$$

Proof: Let $\beta(t) = F(\vec{p} + t\vec{v}) = (f_1(\vec{p} + t\vec{v}), \ldots, f_m(\vec{p} + t\vec{v}))$

Then $\beta'(t) = \left(\frac{d}{dt} f_1(\vec{p} + t\vec{v}), \ldots, \frac{d}{dt} f_m(\vec{p} + t\vec{v})\right)_{\beta(t)}$

$F_*(\vec{v}_p)$ is the initial velocity of the curve $\beta(t)$

so $F_*(\vec{v}_p) = \beta'(0) = \left(\frac{d}{dt} f_1(\vec{p} + t\vec{v})\Big|_{t=0}, \ldots, \frac{d}{dt} f_m(\vec{p} + t\vec{v})\Big|_{t=0}\right)_{\beta(0)}$ QED.

Main idea behind the push forward is to study how a map acts on tangent vectors

- Mappings preserve velocities of curves.

Corollary: Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. If $\beta = F(\alpha)$ is the image of a curve α in $\mathbb{R}^n$, then

$$\beta' = F_*(\alpha')$$

Proof: $\alpha'(t) = \left(\frac{d\alpha_1(t)}{dt}, \dots, \frac{d\alpha_n(t)}{dt}\right)_{\alpha(t)}$

Then $F_*(\alpha'(t)) = (\alpha'(t)[f_1], \dots, \alpha'(t)[f_m])_{F(\alpha(t))}$

$$\alpha'(t)[f_i] = \sum_j \frac{d\alpha_j(t)}{dt} \frac{\partial f_i}{\partial x_j}\Big|_{\alpha(t)} = \frac{d f_i(\alpha(t))}{dt} \text{ (chain rule)}$$

$$\text{so } F_*(\alpha'(t)) = \left(\frac{df_1}{dt}(\alpha(t)), \dots, \frac{df_m}{dt}(\alpha(t))\right)_{F(\alpha(t))}$$

$$= \left(\frac{df_1}{dt}(\alpha(t)), \dots, \frac{df_m}{dt}(\alpha(t))\right)_{\beta(t)} = \beta'(t)$$

Since $\beta(t) = F(\alpha(t)) = (f_1(\alpha(t)), \dots, f_m(\alpha(t)))$

$$\Rightarrow = \left(\sum_i^n \frac{d\alpha_i(t)}{dt} \frac{\partial f_1}{\partial x_i}\Big|_{\alpha(t)}, \dots, \sum_i^n \frac{d\alpha_i(t)}{dt} \frac{\partial f_m}{\partial x_i}\Big|_{\alpha(t)}\right) = \underbrace{J_F(\alpha(t)) \cdot \alpha'(t)}_{\text{Jacobian}} \quad \text{QED.}$$

Let $\{U_j\}$ for $1 \le j \le n$ and $\{\bar{U}_i\}$ for $1 \le i \le n$ be the natural fields of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Corollary: If $F = (f_1, \dots, f_m)$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, then

$$F_*(U_j(\vec{p})) = \sum_{i=1}^m \frac{\partial f_i}{\partial x_j}(\vec{p})\, \bar{U}_i(F(\vec{p}))$$

Proof: We know $F_*(\vec{v}_p) = (\vec{v}_p[f_1], \dots, \vec{v}_p[f_m])_{F(\vec{p})}$

$$F_*(U_j(\vec{p})) = (U_j(\vec{p})[f_1], \dots, U_j(\vec{p})[f_m])_{F(\vec{p})}$$

$$= \left(\frac{\partial f_1}{\partial x_j}\Big|_p, \dots, \frac{\partial f_m}{\partial x_j}\Big|_p \right)_{F(\vec{p})} \quad \text{since } U_j[f_i] = \frac{\partial f_i}{\partial x_j}$$

But $\overline{U}_i(F(\vec{p})) = (0, \dots, 0, \underset{\text{ith place}}{1}, 0, \dots, 0)_{F(\vec{p})}$

$$\text{so } F_*(U_j(\vec{p})) = \sum_{i=1}^{m} \overline{U}_i(F(\vec{p})) \frac{\partial f_i}{\partial x_j}\Big|_{\vec{p}}$$

Since we can think of U_j as $\frac{\partial}{\partial x_j}$ in the $T_p(\mathbb{R}^n)$ space

and $\overline{U}_i$ as $\frac{\partial}{\partial y_i}$ in the $T_{F(p)}(\mathbb{R}^m)$ space

we have exactly what the push forward says.

QED.

Since then we have

$$F_*\left(\frac{\partial}{\partial x_j}\Big|_{\vec{p}} \right) = \sum_{i=1}^{m} \left(\frac{\partial}{\partial y_i}\Big|_{F(p)} \right) \frac{\partial f_i}{\partial x_j}\Big|_{\vec{p}}$$

Jacobian get out from this.

To equivalent.

Example: $F(x_1, x_2) = (\underbrace{e^{x_1+x_2}}_{f_1}, \underbrace{\sin(x_2)}_{f_2}, \underbrace{\cos(x_2)}_{f_3})$ is a mapping from $\mathbb{R}^2 \to \mathbb{R}^3$, with cartesian coordinate (y_1, y_2, y_3) for the image

Find what $\frac{\partial}{\partial x_1}\Big|_{\vec{p}} \mapsto$ and what $\frac{\partial}{\partial x_2}\Big|_{\vec{p}} \longmapsto$ where $\vec{p} = (p_1, p_2)$

Solution: $F_*\left(\frac{\partial}{\partial x_2}\Big|_{\vec{p}}\right) = \frac{\partial f_1}{\partial x_2}\Big|_{\vec{p}} \frac{\partial}{\partial y_1}\Big|_{F(\vec{p})} + \frac{\partial f_2}{\partial x_2}\Big|_{\vec{p}} \frac{\partial}{\partial y_2}\Big|_{F(\vec{p})} + \frac{\partial f_3}{\partial x_2}\Big|_{\vec{p}} \frac{\partial}{\partial y_3}\Big|_{F(\vec{p})}$

$$= e^{x_1+x_2}\Big|_{\vec{p}} \frac{\partial}{\partial y_1}\Big|_{F(\vec{p})} + \cos(x_2)\Big|_{\vec{p}} \frac{\partial}{\partial y_2}\Big|_{F(\vec{p})} - \sin(x_2)\Big|_{\vec{p}} \frac{\partial}{\partial y_3}\Big|_{F(\vec{p})}$$

So $\frac{\partial}{\partial x_2}\Big|_{\vec{p}} \mapsto e^{p_1+p_2} \frac{\partial}{\partial y_1}\Big|_{F(\vec{p})} + \cos(p_2) \frac{\partial}{\partial y_2}\Big|_{F(\vec{p})} - \sin(p_2) \frac{\partial}{\partial y_3}\Big|_{F(\vec{p})}$

where $F(\vec{p}) = (e^{p_1+p_2}, \sin(p_2), \cos(p_2))$ ✓

Example: $F(r, \theta) = (\underbrace{r\cos\theta}_{f_1}, \underbrace{r\sin\theta}_{f_2})$ with cartesian coordinates (y_1, y_2) for the image.

Find $\frac{\partial}{\partial r}\Big|_{\vec{p}} \longmapsto$ and what $\frac{\partial}{\partial \theta}\Big|_{\vec{p}} \longmapsto$ where $\vec{p} = (r_1, \theta_1)$

Solution: $F_*\left(\frac{\partial}{\partial r}\Big|_{\vec{p}}\right) = \frac{\partial f_1}{\partial r}\Big|_{\vec{p}} \frac{\partial}{\partial y_1}\Big|_{F(\vec{p})} + \frac{\partial f_2}{\partial r}\Big|_{\vec{p}} \frac{\partial}{\partial y_2}\Big|_{F(\vec{p})}$

$$= \cos\theta\Big|_{\vec{p}} \frac{\partial}{\partial y_1}\Big|_{F(\vec{p})} + \sin\theta\Big|_{\vec{p}} \frac{\partial}{\partial y_2}\Big|_{F(\vec{p})}$$

So $\frac{\partial}{\partial r}\Big|_{\vec{p}} \longmapsto \cos(\theta_1) \frac{\partial}{\partial y_1}\Big|_{F(\vec{p})} + \sin(\theta_1) \frac{\partial}{\partial y_2}\Big|_{F(\vec{p})}$

where $F(\vec{p}) = (r_1\cos\theta_1, r_1\sin\theta_1)$

Looks familiar?
let $y_1 = x = r\cos\theta$
$y_2 = y = r\sin\theta$
then $\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$

Just as the derivative of a function is used to gain information about the function, the tangent map F_* can be used in the study of a mapping F.

Def: A mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ is regular provided that at every point $\vec{p}$ of $\mathbb{R}^n$ the tangent map F_{*p} is one-to-one.

Since tangent maps are linear transformations, standard results of linear algebra show that the following conditions are equivalent:

(1) F_{*p} is one-to-one

(2) $F_*(\vec{v}_p) = 0 \Rightarrow \vec{v}_p = 0$

(3) The Jacobian matrix of F at $\vec{p}$ has rank n, the dimension of the domain $\mathbb{R}^n$ of F.

The following noteworthy property of linear transformations $T: V \to W$ will be useful in dealing with tangent maps. If the vector spaces V and W have the same dimension, then T is one-to-one iff it is onto, so either property is equivalent to T being a linear isomorphism.

A mapping that has a differentiable inverse mapping is called a diffeomorphism. The results in this section all remain valid when Euclidean spaces $\mathbb{R}^n$ are replaced by open sets of Euclidean spaces, so we can speak of a diffeomorphism from one set to another. [Next is a fundamental result of advanced calculus]

Theorem: Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a mapping between Euclidean spaces of the same dimension. If F_{*p} is one-to-one at a point $\vec{p}$, there is an open set U containing $\vec{p}$ such that F restricted to U is a diffeomorphism of U onto an open set V.

This is called the inverse function theorem since it says that the restricted mapping $U \to V$ has a differentiable mapping $V \to U$.

Ex: (a) Give an example to demonstrate that a one-to-one and onto mapping need not be a diffeomorphism.

$f : \mathbb{R} \to \mathbb{R} \ni f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x} = g(x)$

$g'(x) = \frac{1}{3}(x)^{-\frac{2}{3}}$ so not everywhere differentiable on $\mathbb{R}$ $(x=0)$

(b) Prove that if a one-to-one and onto mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is regular, then it is a diffeomorphism.

For F to be a diffeomorphism, we need F to be bijective and its inverse differentiable.

F is a bijection by assumption and is regular so $F'(x) \neq 0 \ \forall x \in \mathbb{R}^n$.

By the inverse function theorem, for $b = F(a)$, b and $a \in \mathbb{R}^n$; then

$$(F^{-1})'(b) = \frac{1}{F'(a)} \quad \text{since } (F^{-1})'(b) = \frac{1}{F'(F^{-1}(b))}$$

This is clearly well-defined, since F is regular. Thus F is a diffeomorphism.

QED.

Motivations for G.R

Gravitation is a manifestation of space time curvature, and that curvature shows up in the deviation of one geodesic from a nearby geodesic ("relative acceleration of test particles"). The fundamental aspect of G.R for which differential geometry is extremely useful is

- How can one quantify the "separation", and the rate of change" of "separation," of two geodesics" in curved "spacetime".

▸ "separation" between geodesics will mean "vector". But the concept of vector as employed in flat Lorentz spacetime (a bilocal object: point for head and point for tail) must be sharpened up into the local concept of tangent vector, when one passes to curved spacetime. It also reveals how the passage to curved spacetime affects 1-forms and tensors.

Where a geodesic is the curve of shortest length in the given space/geometry that passes through two given points.

Then talk about lecture plan

I. 1 person $\longmapsto$ 2H (1 chapter)

II. 2 person $\longmapsto$ 1H each (1 chapter)

III. 3 person $\longmapsto$ 2H ($\frac{1}{2}$ chapter)