From Harmonic Oscillators to Fields

Quantization of Free Fields

- The classical action for N harmonic oscillators:

$$S[q_i] = \frac{1}{2}\int\left[\sum_{i=1}^{N}\dot{q}_i^2 - \sum_{i,j=1}^{N} M_{ij} q_i q_j\right]dt \qquad (M_{ij} \text{ describes the coupling})$$

Can be reduced to

$$S[\tilde{q}_\alpha] = \frac{1}{2}\int\sum_{\alpha=1}^{N}\left(\dot{\tilde{q}}_\alpha^2 - \omega_\alpha^2 \tilde{q}_\alpha^2\right)dt$$

by decoupling, where ω_α are <u>eigenfrequencies</u> and $\tilde{q}_\alpha$ are the <u>normal modes</u>.

We will henceforth omit the tilde.

- The normal modes q_α are quantized by introducing the operators $\hat{q}_\alpha(t)$, $\hat{p}_\alpha(t)$ and imposing the commutation relations:

$$[\hat{q}_\alpha, \hat{p}_\beta] = i\delta_{\alpha\beta} \qquad [\hat{q}_\alpha, \hat{q}_\beta] = [\hat{p}_\alpha, \hat{p}_\beta] = 0$$

- The Creation and annihilation operators $\hat{a}_\alpha^\pm(t)$ are defined by

$$\hat{a}_\alpha^\pm(t) = \sqrt{\frac{\omega_\alpha}{2}}\left(\hat{q}_\alpha(t) \mp \frac{i}{\omega_\alpha}\hat{p}_\alpha(t)\right)$$

We only need the time-independent version of these: $\hat{a}_\alpha^\pm$

- The vacuum state $|0,\dots,0\rangle$ is the unique common eigenvector of all annihilation operators $\hat{a}_\alpha^-$ with eigenvalue 0, $\hat{a}_\alpha^-|0,\dots,0\rangle = 0$ for $\alpha = 1,\dots,N$
- The state $|n_1, n_2, \dots, n_N\rangle$ with occupation number n_α is defined by

$$|n_1,\dots,n_N\rangle = \left[\prod_{\alpha=1}^{N}\frac{(\hat{a}_\alpha^+)^{n_\alpha}}{\sqrt{n_\alpha!}}\right]|0,0,\dots,0\rangle$$

From Oscillations to Fields

- Now imagine that there is an oscillator at each point in space. The action becomes

$$S[\varphi] = \frac{1}{2}\int dt\left[\int d^3x\,\dot{\varphi}^2(x,t) - \int d^3x\,d^3y\,\varphi(x,t)\varphi(y,t)M(x,y)\right]$$

- The simplest Poincaré-invariant action for a real scalar field is

$$S[\varphi] = \frac{1}{2}\int d^4x\left[\eta^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - m^2\varphi^2\right] \qquad \text{where } \eta \text{ is the Minkowski metr}$$

$$= \frac{1}{2}\int d^3x\,dt\left[\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right]$$

$$\Rightarrow M(x,y) = [\Delta_x + m^2]\,\delta(x-y)$$

- To calculate the equations of motion, take the functional derivative:

$$\frac{\delta S}{\delta\varphi(x,t)} = \ddot{\varphi}(x,t) - \Delta\varphi(x,t) + m^2\varphi(x,t) = 0$$

- To decouple the oscillators φ_x, apply the Fourier transform:

$$\varphi_k(t) \equiv \int \frac{d^3x}{(2\pi)^{3/2}} e^{-ikx} \varphi(x,t)$$

$$\varphi(x,t) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} e^{ikx} \varphi_k(t)$$

- The complex functions $\varphi_k(t)$ are called the <u>modes</u> of the field φ, and have the following properties:
  - $\frac{d^2}{dt^2}\varphi_k(t) + (k^2+m^2)\varphi_k(t) = 0$
  - $\omega_k \equiv \sqrt{k^2+m^2}$
  - $S = \frac{1}{2}\int dt\, d^3k\, (\dot{\varphi}_k \dot{\varphi}_{-k} - \omega_k^2 \varphi_k \varphi_{-k})$

Quantizing Fields in Flat Spacetime

- The Lagrangian is $L[\varphi] = \int \mathcal{L} d^3x$; $\mathcal{L} \equiv \frac{1}{2}\eta^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}m^2\varphi^2$

- The Classical Hamiltonian:

$$H = \int \pi(x,t)\,\dot{\varphi}(x,t)\,d^3x - L \qquad \text{where } \pi(x,t) = \dot{\varphi}(x,t)$$

$$= \frac{1}{2}\int d^3x\,[\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2]$$

- Now Quantize to get

$$\hat{H} = \int d^3k \frac{\omega_k}{2}(\hat{a}_k^- \hat{a}_k^+ + \hat{a}_k^+ a_k^-) = \int d^3k \frac{\omega_k}{2}[2\hat{a}_k^+\hat{a}_k^- + \delta^{(3)}(0)]$$

Mode Expansions

- The mode operator: $\hat{\varphi}_k(t) = \frac{1}{\sqrt{2\omega_k}}(\hat{a}_k^- e^{-i\omega_k t} + \hat{a}_{-k}^+ e^{i\omega_k t})$

- expansion of the field operator:

$$\hat{\varphi}(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}}[\hat{a}_k^- e^{-i\omega_k t + ik\cdot x} + \hat{a}_k^+ e^{i\omega_k t + ik\cdot x}]$$

The Unruh Effect

indler Spacetime

- Consider an object moving with constant acceleration and trajectory $x^\mu(\tau)$ where τ is the proper time measured by the observer.
- We have the following conditions:

$$\star \quad u^\mu u_\mu = 1 \qquad u^\mu \equiv \frac{dx^\mu}{d\tau}$$

$$\star \quad a^\mu \equiv \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$$

$$\star \quad a^\mu a_\mu = -|\vec{a}|^2$$

- Now derive the trajectory $x^\mu(\tau)$ of the accelerated observer.

$\star$ Assume $\vec{a} = (a, 0, 0)$ with $a > 0$

$\star$ that the observer only moves in the x-direction

Then we have

$$\star \quad x(\tau) = x_0 - \frac{1}{a} + \frac{1}{a}\cosh(a\tau)$$

$$\star \quad t(\tau) = t_0 + \frac{1}{a}\sinh(a\tau)$$

For initial conditions $x(0) = \frac{1}{a}$ and $t(0) = 0$, we just have $x^2 - t^2 = \frac{1}{a^2}$

Coordinates in the proper frame

- Coordinates in the proper frame are (τ, ξ) where τ is proper time and ξ is distance measured by the observer

(τ, ξ) is related to (t, x) in the following way:

$$\left.\begin{aligned} t(\tau, \xi) &= x^0(\tau) + s^0_{Lab} = x^0(\tau) + \frac{dx^1(\tau)}{d\tau}\xi \\ x(t, \xi) &= x^1(\tau) + s^1_{Lab} = x^1(\tau) + \frac{dx^0(\tau)}{d\tau}\xi \end{aligned}\right\} \text{For any trajectory } x^{0,1}(\tau)$$

For a uniformly accelerated observer with the initial conditions above, we have:

$$t(\tau, \xi) = \frac{1 + a\xi}{a}\sinh a\tau$$

$$x(\tau, \xi) = \frac{1 + a\xi}{a}\cosh a\tau$$

and

$$\tau(t, x) = \frac{1}{2a}\ln\frac{x + t}{x - t}$$

$$\xi(t, x) = \frac{-1}{a} + \sqrt{x^2 - t^2}$$

The Horizon

An accelerated observer cannot measure distances longer than $\frac{1}{a}$ in the direction opposite to the acceleration.

To see that the line $\xi = -\frac{1}{a}$ is a horizon, consider a line of proper length $\xi = \xi_0 > -\frac{1}{a}$

then $x^2 - t^2 = \text{constant}$ with proper acceleration

$$a_0 \equiv \frac{1}{\sqrt{x^2 - t^2}} = \left(\xi_0 + \frac{1}{a}\right)^{-1}$$

$\Rightarrow$ the worldline $\xi_0 = -\frac{1}{a}$ represents infinite proper acceleration.

Metric of the Rindler Spacetime

The Minkowski metric in proper coordinates is $ds^2 = dt^2 - dx^2 = (1 + a\xi)^2 d\tau^2 - d\xi^2$

To rewrite this metric in a conformally flat form, choose new coordinates $\tilde{\xi}$ such that $d\xi = (1 + a\xi) d\tilde{\xi}$ so that both $d\tau^2$ and $d\tilde{\xi}^2$ have a factor $(1 + a\xi)^2$ in common: $\tilde{\xi} \equiv \frac{1}{a} \ln(1 + a\xi)$. In conformal coordinates $ds^2 = e^{2a\tilde{\xi}}(d\tau^2 - d\tilde{\xi}^2)$ and the relationship between laboratory and conformal coordinates is

$$t(\tau, \tilde{\xi}) = \frac{1}{a} e^{a\tilde{\xi}} \sinh a\tau$$

$$x(\tau, \tilde{\xi}) = \frac{1}{a} e^{a\tilde{\xi}} \cosh a\tau$$

Quantum Fields in the Rindler Spacetime

Consider a massless field in 1+1 dimensional spacetime

The action is $$S[\varphi] = \frac{1}{2} \int g^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} \sqrt{-g}\, d^2x$$

If we replace $g_{\alpha\beta}$ with $\tilde{g}_{\alpha\beta}$, where $\tilde{g}_{\alpha\beta} = \Omega^2(t, x) g_{\alpha\beta}$ then $\sqrt{-g} \to \Omega^2 \sqrt{-g}$ and $g^{\alpha\beta} \to \Omega^{-2} g^{\alpha\beta}$

so Ω^2 cancels.

$\Rightarrow$ We say that a minimally coupled massless scalar field in 1+1 Minkowski spacetime is conformally coupled. This is not so in 3+1 dimensions.

In both laboratory and conformal coordinates, the action is

$$S[\varphi] = \frac{1}{2} \int [(\partial_t \varphi)^2 - (\partial_x \varphi)^2]\, dt\, dx$$

With classical equations of motion:

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = 0 \qquad \frac{\partial^2 \varphi}{\partial \tau^2} - \frac{\partial^2 \varphi}{\partial \tilde{\xi}^2} = 0$$

with general solutions:

$$\varphi(t,x) = A(t-x) + B(t+x) \qquad \varphi(\tau,\tilde{\xi}) = P(\tau-\tilde{\xi}) + Q(\tau+\tilde{\xi})$$

Quantization

- The vaccum state in the laboratory frame $|0_M\rangle$ is the zero eigenvector for all annihilation operators $\hat{a}_k^-$, $\hat{a}_k^-|0_M\rangle = 0 \ \forall k$. with $\hat{\varphi}(t,x) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} \left[e^{-i|k|t + ikx} \hat{a}_k^- + e^{i|k|t - ikx} \hat{a}_k^+\right]$
- The mode expansion in the accelerated frame:

$$\hat{\varphi}(\tau,\tilde{\xi}) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} \left[e^{-i|k|\tau + ik\tilde{\xi}} \hat{b}_k^- + e^{i|k|\tau - ik\tilde{\xi}} \hat{b}_k^+\right]$$

- The vacuum state in the accelerated frame is defined by (Rindler vacuum)

$$\hat{b}|0_R\rangle = 0 \ \forall k$$

Light Cone Mode Expansions

Light Cone Coordinates:

$$\left.\begin{aligned} \bar{u} &\equiv t - x \\ \bar{v} &\equiv t + x \end{aligned}\right\} \text{unaccelerated} \qquad \left.\begin{aligned} u &\equiv \tau - \tilde{\xi} \\ v &\equiv \tau + \tilde{\xi} \end{aligned}\right\} \text{Freely falling} \qquad \left.\begin{aligned} \bar{u} &= -\frac{1}{a} e^{-au} \\ \bar{v} &= \frac{1}{a} e^{av} \end{aligned}\right\} \text{conversion}$$

$$\Rightarrow ds^2 = d\bar{u}\,d\bar{v} = e^{a(v-u)} du\,dv$$

$$\left.\begin{aligned} \frac{\partial^2}{\partial \bar{u}\partial \bar{v}} \varphi(\bar{u},\bar{v}) &= 0 \qquad & \varphi(\bar{u},\bar{v}) &= A(\bar{u}) + B(\bar{v}) \\ \frac{\partial^2}{\partial u \partial v} \varphi(u,v) &= 0 \qquad & \varphi(u,v) &= P(u) + Q(v) \end{aligned}\right\} \text{Field equations \& general solutions}$$

Write the mode expansion $\hat{\varphi}(t,x)$ in its positive & negative ranges:

$$\hat{\varphi}(t,x) = \int_{-\infty}^{0} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} \left[e^{ikt + ikx} \hat{a}_k^- + e^{-ikt - ikx} \hat{a}_k^+\right] + \int_{0}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2k}} \left[e^{-ikt + ikx} \hat{a}_k^- + e^{ikt - ikx} \hat{a}_k^+\right]$$

Then let $\omega = |k|$ to obtain the light cone mode expansion:

$$\hat{\varphi}(\bar{u},\bar{v}) = \int_{0}^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+ + e^{-i\omega\bar{v}} \hat{a}_{-\omega}^- + e^{i\omega\bar{v}} \hat{a}_{-\omega}^+\right]$$

$$\Rightarrow \quad \hat{A}(\bar{u}) = \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_{\omega}^{-} + e^{i\omega\bar{u}} \hat{a}_{\omega}^{+} \right]$$

$$\hat{B}(\bar{v}) = \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{v}} \hat{a}_{-\omega}^{-} + e^{i\omega\bar{v}} \hat{a}_{-\omega}^{+} \right]$$

Similarly for the Rindler frame:

$$\hat{\varphi}(u,v) = \hat{P}(u) + \hat{Q}(v) = \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_{\Omega}^{-} + e^{i\Omega u} \hat{b}_{\Omega}^{+} + e^{-i\Omega v} \hat{b}_{-\Omega}^{-} + e^{i\Omega v} \hat{b}_{-\Omega}^{+} \right]$$

Note: The Rindler mode expansion is only valid within the domain $x > |t|$

The Bogolyubov transformations

We want to find the relationship between $\hat{a}_{\pm\omega}^{\pm}$ and $\hat{b}_{\pm\Omega}^{\pm}$.

We have: $\quad \hat{\varphi}(u,v) = \hat{A}(\bar{u}(u)) + \hat{B}(\bar{v}(v)) = \hat{P}(u) + \hat{Q}(v)$

$$\Rightarrow \hat{A}(\bar{u}) = \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega\bar{u}} \hat{a}_{\omega}^{-} + e^{i\omega\bar{u}} \hat{a}_{\omega}^{+} \right]$$

$$= \hat{P}(u) = \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_{\Omega}^{-} + e^{i\Omega u} \hat{b}_{\Omega}^{+} \right]$$

Apply the Fourier transform:

$$\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{P}(u) = \frac{1}{\sqrt{2|\Omega|}} \begin{cases} \hat{b}_{\Omega}^{-}, & \Omega > 0 \\ \hat{b}_{|\Omega|}^{+}, & \Omega < 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{A}(\bar{u}) = \int_0^{\infty} \frac{d\omega}{\sqrt{2\omega}} \int_{-\infty}^{+\infty} \frac{du}{2\pi} \left[e^{i\Omega u - i\omega\bar{u}} \hat{a}_{\omega}^{-} + e^{i\Omega u + i\omega\bar{u}} \hat{a}_{\omega}^{+} \right]$$

$$\equiv \int_0^{\infty} \frac{d\omega}{\sqrt{2\omega}} \left[F(\omega,\Omega) \hat{a}_{\omega}^{-} + F(-\omega,\Omega) \hat{a}_{\omega}^{+} \right]$$

Where $F(\omega,\Omega) = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{i\Omega u - i\omega\bar{u}}$

So we have

$$\hat{b}_{\Omega}^{-} = \int_0^{\infty} d\omega \left[\alpha_{\omega\Omega} \hat{a}_{\omega}^{-} + \beta_{\omega\Omega} \hat{a}_{\omega}^{+} \right] \quad \text{where} \quad \alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(\omega,\Omega) \text{ and } \beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(-\omega,\Omega)$$

The other Bogolyubov transformation are done in a similar way

Note: $\quad F^{*}(\omega,\Omega) = F(-\omega,-\Omega)$

Density of Particles

$|0_M\rangle$ + $|0_R\rangle$ correspond to the operators $\hat{a}_\omega$ and $\hat{b}_\omega$. The a-vacuum state has b-particles + vice versa. We can compute the density of b-particles in the a-vacuum state:

$$\hat{N}_\Omega = \hat{b}^+_\Omega \hat{b}^-_\Omega$$

$$\Rightarrow \langle \hat{N}_\Omega \rangle \equiv \langle 0_M | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_M \rangle$$

$$= \langle 0_M | \int d\omega [\alpha^*_{\omega\Omega} \hat{a}^+_\omega + \beta^*_{\omega\Omega} \hat{a}^-_\omega] \int d\omega' [\alpha_{\omega'\Omega} \hat{a}^-_{\omega'} + \beta_{\omega'\Omega} \hat{a}^+_{\omega'}] | 0_M \rangle$$

$$= \int d\omega |\beta_{\omega\Omega}|^2$$ (Mean number of particles in the accelerated frame)

(here I skipped a lot of steps) $$= \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1} \delta(0)$$

$\Rightarrow$ The mean density of particles in the mode with momentum Ω is

$$n_\Omega = \left[\exp\left(\frac{2\pi\Omega}{a}\right) - 1\right]^{-1}$$

The Unruh Temperature

A massless particle with momentum Ω has energy $\epsilon = |\Omega|$ so the formula n_Ω is equivalent to the Bose-Einstein distribution

$$n(\epsilon) = \left[\exp\left(\frac{\epsilon}{T}\right) - 1\right]^{-1}$$

Where T is the Unruh Temperature $T \equiv \frac{a}{2\pi}$

*An accelerated particle detector behaves as though it were placed in a thermal bath with temperature T. This is the Unruh effect.

The Hawking Radiation

Scalar field in Black Hole spacetime

- Consider a scalar field in the presence of a single, nonrotating black hole with mass M
This spacetime is described by the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - 2M/r} - r^2(d\theta^2 + d\varphi^2 \sin^2\theta)$$

* The horizon corresponds to $r = 2M$
* (t, r) has its normal interpretation of time and space only for $r > 2M$
* We will again restrict our attention to $1+1$ spacetime, where

$$ds^2 = g_{ab}dx^a dx^b \qquad x^0 \equiv t, \quad x^1 \equiv r$$

with the reduced metric

$$g_{ab} = \begin{bmatrix} 1 - 2M/r & 0 \\ 0 & -(1 - 2M/r)^{-1} \end{bmatrix}$$

and action:

$$S[\varphi] = \frac{1}{2}\int g^{ab}\varphi_{,a}\varphi_{,b}\sqrt{-|g_{ab}|}\, d^2x$$

Put the metric in conformally flat form by making the coordinate change $r \to r^*$ where r^* is such that

$$dr = \left(1 - \frac{2M}{r}\right)dr^*$$

$$\Rightarrow r^*(r) = r - 2M + 2M \ln\left(\frac{r}{2M} - 1\right)$$

$$\Rightarrow ds^2 = \left(1 - \frac{2M}{r}\right)[dt^2 - dr^{*2}]$$ (we don't need an explicit formula for $r(r^*)$)

* $r^*(r)$ is only defined for $r > 2M$ and is called the tortoise coordinate.

$$S[\varphi] = \frac{1}{2}\int [(\partial_t\varphi)^2 - (\partial_{r^*}\varphi)^2]\, dt\, dr^*$$

$$\varphi(t, r^*) = P(t - r^*) + Q(t + r^*)$$

In light cone coordinates: $u \equiv t - r^*$ $\quad v \equiv t + r^*$, the metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right)du\, dv$$

Note: (t, r^*) coincides asymtotically with Minkowski coordinates for $r^* \to +\infty$

The Kruskal Coordinates

The singularity in the Minkowski coordinates is a "coordinate singularity." An observer crossing the horizon line $r = 2M$ will observe normal spacetime. The Kruskal frame describes space from the perspective of a freely falling observer.

We write Kruskal light-frame coordinates:

$$\bar{u} = -4M \exp\left(-\frac{u}{4M}\right) \qquad \bar{v} = 4M \exp\left(\frac{v}{4M}\right)$$

where $-\infty < \bar{u} < 0 \quad , \quad 0 < \bar{v} < +\infty$

$$t = 2M \ln\left(-\bar{v}/\bar{u}\right)$$

$$\exp\left(-r^{*}/2M\right) = -\frac{16M^2}{\bar{u}\bar{v}}$$

The Black Hole horizon $r = 2M$ corresponds to the lines $\bar{u} = 0$, $\bar{v} = 0$. For the Metric we have:

$$ds^2 = -\frac{16M^2}{\bar{u}\bar{v}}\left(1 - \frac{2M}{r}\right) d\bar{u}\, d\bar{v}$$

$$= \frac{2M}{r} \exp\left(1 - \frac{r}{2M}\right) d\bar{u}\, d\bar{v}$$

So when $r = 2M$, $ds^2 = d\bar{u}\, d\bar{v}$, the same as in Minkowski spacetime.

We will use $(\bar{u}, \bar{v})$ to refer to freely falling observers and (u, v) to describe accelerated frames

Field Quantization

Now we will quantize the field $\varphi(x)$ in both the Kruskal frame and the tortoise frame

Lightcone Mode expansion for tortoise coordinates:

$$\hat{\varphi}(u,v) = \int_0^{+\infty} \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega u} \hat{b}_{\Omega}^{-} + \text{H.c.} + e^{-i\Omega v} \hat{b}_{-\Omega}^{-} + \text{H.c.} \right]$$

where H.c. = Hermitian conjugate terms.

As before, $\hat{b}^{\pm}_{\pm\Omega}$ corresponds to a stationary observer.

Lightcone Mode expansion for Kruskal coordinates:

$$\hat{\varphi}(\bar{u},\bar{v}) = \int_0^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega \bar{u}} \hat{a}_{\omega}^{-} + \text{H.c.} + e^{-i\omega \bar{v}} \hat{a}_{-\omega}^{-} + \text{H.c.} \right]$$

where this time $\hat{a}^{\pm}_{\pm\omega}$ corresponds to an observer freely falling into a black hole.

=> We have two vacuum states: $|0_K\rangle$ and $|0_T\rangle$ for each set of coordinates.

$|0_T\rangle$ is also called the Boulware vacuum.

Similar to what we did with Unruh effect, we can use the Bogolyubov transformations to go between $\hat{a}^{\pm}_{\pm\omega}$ & $\hat{b}^{\pm}_{\pm\Omega}$.

*Note that the comparison of Rindler and Schwarzschild spacetimes is a good analogy only for a conformally coupled field in 1+1 toy model spacetime.

The Hawking Temperature:

For observers at $r >> 2M$ away from a blackhole, the ambient vacuum state is the Minkowski state $|0_M\rangle$ which is approximately the same as the Boulware vacuum $|0_T\rangle$

We can use our analogy between Rindler & Schwartzschild spacetimes to require $a = \frac{1}{4M}$

=> An observer at $r >> 2M$ detects a thermal spectrum of particles with temperature

$$T_H = \frac{1}{8\pi M} \quad , \text{ the Hawking Temperature.}$$

*The closer you get to the black hole, the higher the observed temperature due to the inverse gravitational redshift.

The density of the observed particles with energy $\epsilon = k$ is (momentum)

$$n_\epsilon = \left[\exp\left(\frac{\epsilon}{T_H}\right) - 1\right]^{-1}$$

*Note that if we set $m^2 + k^2 = \epsilon^2$, the particle production is only significant for particles with mass $m < T_H$ & T_H for a plausible actual black hole is very small. I don't think Hawking radiation has ever actually been observed.

References:

- Introduction to Quantum Fields in Classical Backgrounds
  By V.F. Mukhanov and S. Winitzki
- Quantum Field theory in Curved Spacetime
  By Robert M. Wald.

Goals for future study: Hawking Radiation & the Unruh effect in 3+1 dimensions.