Helicity, chirality, and the Dirac equation in the non-relativistic limit

Logan Meredith

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Abstract

The Dirac equation describes spin-1/2 particles with a consideration for the effects of special relativity. In this paper, we explore two major emergent results of the Dirac equation. First, we see how the notions of helicity and chirality arise from the Dirac equation, and exactly correspond to one another in the massless limit. Second, we verify that the Dirac equation is consistent with the Schrödinger equation in the non-relativistic limit, both for a free particle and for a charged particle in an external magnetic field.

1 Introduction

The Dirac equation, named after Paul Dirac, represented an attempt to incorporate the effects of special relativity into quantum mechanics, and was introduced in 1928 [2]. The result was a wave equation describing the relativistic behavior of spin-1/2 particles, such as electrons and neutrinos. A major departure from previous quantum theories, Dirac's equation describes particles with 4-component spinors or bispinors, rather than scalar wavefunctions. Furthermore, the Dirac equation predicted the existence of antimatter many years before it was experimentally observed.

In this paper, we provide descriptions of two results of the Dirac equation. In the first section, we explore the topics of chirality and helicity. Chirality is an inherent property of particles, whereas helicity of a particle depends on the particle's momentum. However, in the massless limit, the Dirac equation shows that a particle of positive helicity has positive chirality, and vice versa. In the second section, we take the non-relativistic limit of the Dirac equation and show that it reduces to the Schrödinger equation, which describes particles in the non-relativistic regime.

2 Helicity and chirality

In this section, we define helicity and chirality of particles described by the Dirac equation. In addition, we use the Dirac equation to show how massless particles of a given chiral handedness express helicity of the same handedness.

2.1 The helicity operator

The Dirac Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \tag{1}$$

does not necessarily commute with orbital angular momentum or spin angular momentum, but with total angular momentum. When the particle is at rest, however, $\mathbf{p} = 0$, and so

$$[S_i, H] = [\tilde{\alpha}_i/2, H] = -i\epsilon_{ijk}\alpha_j p_k = 0. \tag{2}$$

The Dirac Hamiltonian does commute with p. Furthermore, since p commutes with S, $\mathbf{S} \cdot \mathbf{p}$ also commutes with H. Normalizing over $\mathbf{p}$, we define the helicity operator as

$$h = \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|}, \tag{3}$$

which is necessarily a constant of motion. Physically, the helicity operator can be thought of as a projection operator of spin along the direction of motion.

Note that

$$h^2 = \frac{(\mathbf{S} \cdot \mathbf{p})(\mathbf{S} \cdot \mathbf{p})}{|\mathbf{p}|^2} = \frac{(\tilde{\boldsymbol{\alpha}} \cdot \mathbf{p})(\tilde{\boldsymbol{\alpha}} \cdot \mathbf{p})}{4|\mathbf{p}|^2} = \frac{1}{4}\frac{|\mathbf{p}|^2}{|\mathbf{p}|^2}\mathbb{I} = \frac{1}{4}\mathbb{I}. \tag{4}$$

This implies that the eigenvalues of h are $\pm 1/2$.

2.2 The Dirac equation in the massless limit

We will begin our analysis in this section with massive particles and ultimately look at the massless limit. The free Dirac equation gives

$$(\gamma^\mu p_\mu - m)u(p) = 0, \tag{5}$$

where u is a 4-component spinor. Write

$$u(p) = \begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix}. \tag{6}$$

Then the Dirac equation can be expanded to yield

$$\begin{pmatrix} (p^0 - m)\mathbb{I} & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & -(p^0 + m)\mathbb{I} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0, \tag{7}$$

which, explicitly, gives

$$\begin{aligned} p^0 u_1 - \boldsymbol{\sigma} \cdot \mathbf{p} u_2 &= m u_1, \\ p^0 u_2 - \boldsymbol{\sigma} \cdot \mathbf{p} u_1 &= -m u_2. \end{aligned}$$

Taking the sum and the difference of these equations, we get

$$(p^0 - \boldsymbol{\sigma}\cdot\mathbf{p})(u_1+u_2) = m(u_1-u_2),$$
$$(p^0 + \boldsymbol{\sigma}\cdot\mathbf{p})(u_1-u_2) = m(u_1+u_2).$$

By defining

$$u_l = \frac{1}{2}(u_1-u_2), \tag{8}$$
$$u_r = \frac{1}{2}(u_1+u_2), \tag{9}$$

our equations can be rewritten as

$$(p^0 - \boldsymbol{\sigma}\cdot\mathbf{p})u_r = mu_l,$$
$$(p^0 + \boldsymbol{\sigma}\cdot\mathbf{p})u_l = mu_r.$$

These equations are coupled via the mass term. By letting mass go to zero, we have the uncoupled equations

$$p^0 u_r = \boldsymbol{\sigma}\cdot\mathbf{p}u_r, \tag{10a}$$
$$p^0 u_l = -\boldsymbol{\sigma}\cdot\mathbf{p}u_l. \tag{10b}$$

Since mass is a Lorentz scalar, these equations are Lorentz covariant. However, they are not invariant under parity or space reflections. Equations (10) are known as the Weyl equations [3].

Multiplying both sides of (10) by $p^0 = \boldsymbol{\sigma}\cdot\mathbf{p}$, we find

$$\left(p^0\right)^2 u_r = (\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p})u_r = \mathbf{p}^2 u_r,$$
$$\left(p^0\right)^2 u_l = \mathbf{p}^2 u_l.$$

For a nontrivial solution, we require

$$\left(p^0\right)^2 - \mathbf{p}^2 = 0, \tag{11}$$

which we recognize as Einstein's equation for massless particles. It implies that $p^0 = \pm|\mathbf{p}|$. For $p^0 = +|\mathbf{p}|$, which corresponds to positive energy solutions, we have

$$\frac{1}{|\mathbf{p}|}\boldsymbol{\sigma}\cdot\mathbf{p}u_r = u_r, \tag{12a}$$
$$\frac{1}{|\mathbf{p}|}\boldsymbol{\sigma}\cdot\mathbf{p}u_l = -u_l. \tag{12b}$$

Since $\boldsymbol{\sigma}/2$ is the spin operator for 2-component spinors, (12) imply that u_r, u_l are eigenvectors of the helicity operator h with eigenvalues $+1/2$ and $-1/2$, respectively. We call the particle described by u_r right-handed, and likewise we call the particle described by u_l left-handed. As examples, it has been experimentally determined that electron neutrinos are left-handed particles, whereas electron antineutrinos are right-handed.

2.3 On chirality

The normalized solutions to the massless Dirac equation

$$\not{p}u(p) = 0 = \not{p}v(p) \tag{13}$$

are given by

$$u(p) = \sqrt{\frac{E}{2}}\begin{pmatrix} \tilde{u} \\ \frac{1}{E}\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{u}\end{pmatrix} = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} \tilde{u} \\ \frac{1}{|\mathbf{p}|}\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{u}\end{pmatrix}, \tag{14a}$$

$$v(p) = \sqrt{\frac{E}{2}}\begin{pmatrix} \frac{1}{E}\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{v} \\ \tilde{v}\end{pmatrix} = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} \frac{1}{|\mathbf{p}|}\boldsymbol{\sigma}\cdot\mathbf{p}\tilde{v} \\ \tilde{v}\end{pmatrix}, \tag{14b}$$

where u denotes the positive energy solutions and v denotes the negative energy solutions. Evidently, if u is a solution to (13), so is $\gamma_5 u$, where we define

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{15}$$

Since $\gamma_5^2 = \mathbb{I}$, the eigenvalues of γ_5 are ± 1. We say that eigenvectors of γ_5 with eigenvalue $+1$ have right-handed or positive chirality. Likewise, we say that eigenvectors with eigenvalue -1 have left-handed or negative chirality.

For a general spinor u, we can recover the right-handed and left-handed chiral components by using the chiral projection operators

$$P_R \equiv \frac{\mathbb{I}+\gamma_5}{2}, \tag{16a}$$

$$P_L \equiv \frac{\mathbb{I}-\gamma_5}{2}. \tag{16b}$$

These projection operators have the following properties:

$$\begin{aligned} P_R^2 &= P_R, & P_L^2 &= P_L, \\ P_R P_L &= P_L P_R = 0, & P_R + P_L &= \mathbb{I}. \end{aligned}$$

Together, these properties imply that any 4-component spinor can be uniquely decomposed into right-handed and left-handed components.

The right-handed and left-handed chiral components of the normalized solutions (14) can be explicitly written as

$$u_R = P_R u = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} \frac{1}{2}\left(\mathbb{I}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{u} \\ \frac{1}{2}\left(\mathbb{I}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{u}\end{pmatrix}, \tag{17a}$$

$$u_L = P_L u = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} \frac{1}{2}\left(\mathbb{I}-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{u} \\ -\frac{1}{2}\left(\mathbb{I}-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{u}\end{pmatrix}, \tag{17b}$$

$$v_R = P_R v = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} \frac{1}{2}\left(\mathbb{I}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{v} \\ \frac{1}{2}\left(\mathbb{I}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{v}\end{pmatrix}, \tag{17c}$$

$$v_L = P_L v = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} -\frac{1}{2}\left(\mathbb{I}-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{v} \\ \frac{1}{2}\left(\mathbb{I}-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{v}\end{pmatrix}. \tag{17d}$$

Note that 2-component spinors of the form

$$\chi^{(\pm)} = \frac{1}{2}\left(\mathbb{I} \pm \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{\chi}$$

have definite helicity, so that

$$\begin{aligned} h\chi^{(\pm)} &= \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2|\mathbf{p}|}\frac{1}{2}\left(\mathbb{I} \pm \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{\chi} \\ &= \pm\frac{1}{4}\left(\mathbb{I} \pm \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\tilde{\chi} \\ &= \pm\frac{1}{2}\chi^{(\pm)}. \end{aligned}$$

This means that 4-component spinors with positive chirality are defined by 2-component spinors with positive helicity, and similarly, 4-component spinors with negative chirality are defined by 2-component spinors with negative helicity. Hence we can write

$$u_R = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} u^{(+)} \\ u^{(+)} \end{pmatrix}, \tag{18a}$$

$$u_L = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} u^{(-)} \\ -u^{(-)} \end{pmatrix}, \tag{18b}$$

$$v_R = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} v^{(+)} \\ v^{(+)} \end{pmatrix}, \tag{18c}$$

$$v_L = \sqrt{\frac{|\mathbf{p}|}{2}}\begin{pmatrix} -v^{(-)} \\ v^{(-)} \end{pmatrix}. \tag{18d}$$

Recall the notation

$$\sigma^\mu = (\mathbb{I}, \boldsymbol{\sigma}), \quad \tilde{\sigma}^\mu = (\mathbb{I}, -\boldsymbol{\sigma}),$$

and define

$$\hat{p}_\mu = (1, -\hat{\mathbf{p}}) = \left(1, -\frac{\mathbf{p}}{|\mathbf{p}|}\right).$$

Now define the helical projection operators

$$P^{(+)} = \frac{1}{2}\tilde{\sigma}^\mu \hat{p}_\mu, \tag{19a}$$

$$P^{(-)} = \frac{1}{2}\sigma^\mu \hat{p}_\mu. \tag{19b}$$

These projection operators have the following properties:

$$\begin{aligned} \left(P^{(\pm)}\right)^2 = P^{(\pm)}, \quad P^{(+)} + P^{(-)} = \mathbb{I}, \\ P^{(+)}P^{(-)} = P^{(-)}P^{(+)} = 0. \end{aligned}$$

Hence any 2-component spinor can be uniquely decomposed into components of these operators, which project into spaces of positive and negative helicity. This notion can be generalized to 4-component spinors by defining the following helical projection operator:

$$P^{(\pm)}_{4\times 4} = \begin{pmatrix} P^{(\pm)} & 0 \\ 0 & P^{(\pm)} \end{pmatrix}. \tag{20}$$

It is trivial to check that

$$[P_{R,L}, P^{(\pm)}_{4\times 4}] = 0,$$

which implies that spinors can be simultaneous eigenstates of chirality and helicity in the massless limit.

We have seen that, in the massless limit, particles of positive chirality have positive helicity, and likewise, particles of negative chirality have negative helicity. But this is only true for massless particles. Chirality can be thought of as an inherent trait of particles, whereas helicity depends on the momentum of a particle. For massive particles, it is possible to Lorentz boost to different frames of reference to change helicity. But this is not true for massless particles; hence chirality corresponds exactly to helicity for massless particles.

2.4 Properties of eigenstates of the helicity projections

In this section, we will derive properties of right-handed spinors and state the analogous properties of left-handed spinors, since their derivation is nearly identical.

Write positive and negative energy solutions to the massless Dirac equation with right-handed chirality as

$$u^{(+)} = \frac{1}{2}(\mathbb{I} + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\tilde{u},$$
$$v^{(+)} = \frac{1}{2}(\mathbb{I} + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\tilde{v}.$$

Choose

$$\tilde{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the positive and negative energy solutions can be written explicitly as

$$u^{(+)}(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p_3)}} \begin{pmatrix} |\mathbf{p}| + p_3 \\ p_1 + ip_2 \end{pmatrix},$$
$$v^{(+)}(\mathbf{p}) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p_3)}} \begin{pmatrix} p_1 - ip_2 \\ |\mathbf{p}| - p_3 \end{pmatrix}.$$

It is simple to check that these spinors satisfy

$$u^{(+)\dagger}u^{(+)} = v^{(+)\dagger}v^{(+)} = 1,$$
$$u^{(+)\dagger}(\mathbf{p})v^{(+)}(-\mathbf{p}) = v^{(+)\dagger}(-\mathbf{p})u^{(+)}(\mathbf{p}) = 0,$$
$$u^{(+)}u^{(+)\dagger} = v^{(+)}v^{(+)\dagger} = \frac{1}{2}(\mathbb{I} + \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}).$$

The 4-component spinors defined by (18) further satisfy

$$u_R^\dagger u_R = v_R^\dagger v_R = |\mathbf{p}|, \tag{21a}$$

$$u_R^\dagger(\mathbf{p}) v_R(-\mathbf{p}) = v_R^\dagger(-\mathbf{p}) u_R(\mathbf{p}) = 0, \tag{21b}$$

$$u_R u_R^\dagger = v_R v_R^\dagger = \frac{|\mathbf{p}|}{2} \begin{pmatrix} P^{(+)} & P^{(+)} \\ P^{(+)} & P^{(+)} \end{pmatrix}. \tag{21c}$$

Note that, with $p_0 = |\mathbf{p}|$, we can write

$$\begin{aligned} \frac{1}{4}\not{p}\gamma^0(\mathbf{I} + \gamma_5) &= \frac{1}{4} \begin{pmatrix} p_0 & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -p_0 \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix} \\ &= \frac{|\mathbf{p}|}{4} \begin{pmatrix} \mathbb{I} & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix} \\ &= \frac{|\mathbf{p}|}{2} \begin{pmatrix} P^{(+)} & P^{(+)} \\ P^{(+)} & P^{(+)} \end{pmatrix}. \end{aligned}$$

Hence (21c) can be rewritten as

$$u_R u_R^\dagger = v_R v_R^\dagger = \frac{1}{4}\not{p}\gamma^0(\mathbb{I} + \gamma_5). \tag{22}$$

The analogous relation for left-handed spinors, which we will not derive here, is given by

$$u_L u_L^\dagger = v_L v_L^\dagger = \frac{1}{4}\not{p}\gamma^0(\mathbb{I} - \gamma_5). \tag{23}$$

3 Reduction to the Schrödinger equation

In this section, we consider the free Dirac equation and the Dirac equation for a charged particle in an external magnetic field and show that they reduce to Schrödinger equations describing the same physical systems.

3.1 Free particles

Recall that positive and negative energy solutions to the massive Dirac equation have the form

$$u = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \tilde{u} \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\tilde{u} \end{pmatrix} \equiv \begin{pmatrix} u_{\mathrm{L}} \\ u_{\mathrm{S}} \end{pmatrix}, \tag{24a}$$

$$v = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\tilde{v} \\ \tilde{v} \end{pmatrix} \equiv \begin{pmatrix} v_{\mathrm{S}} \\ v_{\mathrm{L}} \end{pmatrix}. \tag{24b}$$

Note that

$$u_{\mathrm{S}} = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} u_{\mathrm{L}}, \tag{25a}$$

$$v_{\mathrm{S}} = -\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} v_{\mathrm{L}}. \tag{25b}$$

In the non-relativistic limit, $|\mathrm{p}| \ll m$, and so $u_{\mathrm{S}} \ll u_{\mathrm{L}}$ and $v_{\mathrm{S}} \ll v_{\mathrm{L}}$. Hence the subscript S refers to the small component and the subscript L refers to the large component.

The positive energy solutions satisfy

$$\begin{pmatrix} m\mathbb{I} & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & m\mathbb{I} \end{pmatrix} \begin{pmatrix} u_{\mathrm{L}} \\ u_{\mathrm{S}} \end{pmatrix} = E \begin{pmatrix} u_{\mathrm{L}} \\ u_{\mathrm{S}} \end{pmatrix}, \tag{26}$$

which can be expanded to yield

$$\boldsymbol{\sigma}\cdot\mathbf{p}u_{\mathrm{S}} = (E-m)u_{\mathrm{L}}, \tag{27a}$$
$$\boldsymbol{\sigma}\cdot\mathbf{p}u_{\mathrm{L}} = (E+m)u_{\mathrm{S}}. \tag{27b}$$

By substituting (25a) into (27a), we find that

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}u_{\mathrm{L}} = (E-m)u_{\mathrm{L}}. \tag{28}$$

But since $|\mathbf{p}| \ll m$, $E \approx m$, and so we have

$$\frac{\mathbf{p}^2}{2m}u_{\mathrm{L}} = (E-m)u_{\mathrm{L}} = E_{\mathrm{NR}}u_{\mathrm{L}}, \tag{29}$$

where we use the fact that $E_{\mathrm{NR}} \equiv E - m$ is the non-relativistic energy of the particle. Note that (29) is the time-independent Schrödinger equation for a free particle. We require the Dirac equation, a relativistic theory, to reduce to a non-relativistic theory in the non-relativistic theory, and so this result is as expected.

3.2 Charged particles in an external magnetic field

We shall couple a charged particle to a field through a minimal coupling. Take

$$p_\mu \to p_\mu - eA_\mu,$$

where e is the charge and A is the 4-potential. Since the coordinate representation of the momentum is $p_\mu = i\partial_\mu$, we have

$$\partial_\mu \to \partial_\mu + ieA_\mu.$$

The positive energy Dirac equation under this transformation is therefore

$$(\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A}) + \beta m)u = Eu,$$

which can be expanded to yield

$$\begin{pmatrix} m\mathbb{I} & \boldsymbol{\sigma}\cdot(\mathbf{p}-e\mathbf{A}) \\ \boldsymbol{\sigma}\cdot(\mathbf{p}-e\mathbf{A}) & m\mathbb{I} \end{pmatrix} \begin{pmatrix} u_{\mathrm{L}} \\ u_{\mathrm{S}} \end{pmatrix} = E \begin{pmatrix} u_{\mathrm{L}} \\ u_{\mathrm{S}} \end{pmatrix}. \tag{30}$$

This equation results in two equations given by

$$\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})u_{\mathrm{S}} = (E - m)u_{\mathrm{L}}, \tag{31a}$$

$$\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})u_{\mathrm{L}} = (E + m)u_{\mathrm{S}}. \tag{31b}$$

From (31b), we have

$$u_{\mathrm{S}} = \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})}{E + m} u_{\mathrm{L}} \approx \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})}{2m} u_{\mathrm{L}}, \tag{32}$$

in the non-relativistic limit. Substituting (32) into (31a) yields

$$[\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})] \frac{\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})}{2m} u_{\mathrm{L}} \approx (E - m)u_{\mathrm{L}}. \tag{33}$$

Let us compute the products that appear in (33) to reduce notation. Note that

$$\begin{aligned}[\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A})]^2 &= (\mathbf{p} - e\mathbf{A})^2 + i\boldsymbol{\sigma} \cdot [(\mathbf{p} - e\mathbf{A}) \times (\mathbf{p} - e\mathbf{A})] \\ &= (\mathbf{p} - e\mathbf{A})^2 - ie\boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p}).\end{aligned}$$

Furthermore, we calculate

$$\begin{aligned}(\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p})_i &= \epsilon_{ijk}(p_j A_k + A_j p_k) \\ &= \epsilon_{ijk}(p_j A_k - A_k p_j) \\ &= \epsilon_{ijk}[p_j, A_k] \\ &= -i\epsilon_{ijk}[\nabla_j, A_k] \\ &= -i(\boldsymbol{\nabla} \times \mathbf{A})_i \\ &= -iB_i,\end{aligned}$$

where $\mathbf{B}$ is the magnetic field. Hence (33) can be rewritten as

$$\frac{1}{2m}\left[(\mathbf{p} - e\mathbf{A})^2 - e\boldsymbol{\sigma} \cdot \mathbf{B}\right] u_{\mathrm{L}} = E_{\mathrm{NR}} u_{\mathrm{L}}. \tag{34}$$

We recognize this as the Schrödinger equation for interaction with a magnetic dipole. The magnetic dipole moment operator is

$$\boldsymbol{\mu} = \frac{e}{2m}\boldsymbol{\sigma} = g\frac{e}{2m}\mathbf{S}, \tag{35}$$

where $g = 2$ since $\mathbf{S} = \boldsymbol{\sigma}/2$.

Since this is a minimal coupling, we ignore higher-order effects. These are considered further in quantum electrodynamics. These effects tend to slightly change the g-factor for elecrons. Other particles that are not pointlike, such as protons, can have wildly differing g-factors, resulting in anomalous magnetic dipole moments. These moments can be accomodated through a non-minimal coupling with an interaction Hamiltonian

$$H_{\mathrm{I}} = \frac{e\kappa}{2m}\sigma^{\mu\nu}F_{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and κ is the anomalous magnetic dipole moment. This is known as the Pauli interaction.

Notes

This paper represents a reproduction of my lecture notes on the properties of the Dirac equation, which are derived from Section 3 of Das' book on quantum field theory [1]. I have cut down much of the material and added some useful references to get a better picture of the utility and historical context of the content. Consider the text of this paper to be a script and the equations to be things that should be drawn on the chalkboard during the lecture.

References

[1] Ashok Das. *Lectures on quantum field theory*. World Scientific, 2008.

[2] Paul AM Dirac. The quantum theory of the electron. *Proc. R. Soc. Lond. A*, 117(778):610–624, 1928.

[3] Hermann Weyl. Gravitation and the electron. *Proceedings of the National Academy of Sciences*, 15(4):323–334, 1929.