Solving the Left Hand Side of the Einstein Equations for 'Spatially Identical' Diagonal Metrics

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Abstract

Given certain symmetries present in a covariant metric, it is possible to derive a formula for a general case of the Ricci Tensor components and the Ricci Scalar. The following is based off of my second Kapitza lecture, and supplemented by my experimentation with personally written computational techniques that can solve for the left hand side of the Einstein equations.

1 Introduction

The crux of general relativity can be summarized by the famous Einstein equations. In essence this collection of equations relates the shape of spacetime to the distribution of matter and energy. Matter-energy 'directs' the curvature of space-time, and space-time 'directs' the movement of matter-energy.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}$$

$R_{\mu\nu} \equiv$ Ricci Tensor $\mathcal{R} \equiv$ Ricci Scalar $\Lambda \equiv$ Cosmological Constant $g_{\mu\nu} \equiv$ Covariant Metric

$G \equiv$ Gravitational Constant $T_{\mu\nu} \equiv$ Energy Momentum Stress Tensor

The above equation is actually a combination of 16 different equations. However, there are really only 10 equations since the above tensors are all symmetric. With the exception of the cosmological constant, Λ, the entire left hand side (LHS) of the above expression can be computed if given a covariant metric, $g_{\mu\nu}$.

The process of computing the LHS of the Einstein equations can generally be broken down into three steps:

1. Calculate the Christoffel Symbols

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha} \left\{ \frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} \tag{1}$$

The Christoffel Symbols can be thought of as a combination of four matrices (mainly over β) that encode the correction terms for tensor derivatives. The Christoffel Symbol itself is not a tensor.

2. Calculate the components of the Ricci Tensor

$$R_{\mu\nu} = \frac{\partial}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu} - \frac{\partial}{\partial x^{\nu}}\Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{2}$$

(The Rieman Tensor is a rank 4 tensor shown here as the right hand side. However, given the Einstein summation notation, the indices α and β must be summed over, and the Riemann Tensor becomes the Ricci Tensor. This is a higher dimensional analog of the trace.)

3. Calculate the Ricci Scalar

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} \tag{3}$$

The Ricci Scalar is computed, simply by contracting the contravariant metric witht the Ricci Tensor.

While it is possible to solve for an individual $\mu\nu$ component of the Ricci Tensor (and therefore a large part of the respective Einstein Equation), the Ricci Scalar requires all elements of the Ricci Tensor in order to be computed. Thus, most of the time it makes sense to calculate all components of the Ricci Tensor to start with.

2 Solving for the LHS with a General Case Metric

$$g_{\mu\nu} = \begin{bmatrix} f_0(\vec{x}) & 0 & 0 & 0 \\ 0 & f_1(\vec{x}) & 0 & 0 \\ 0 & 0 & f_2(\vec{x}) & 0 \\ 0 & 0 & 0 & f_3(\vec{x}) \end{bmatrix}$$

2.1 Diagonal Metrics

If the metric can be expressed diagonally (through a change of coordinate basis) then several key simplifications can be made to compute the resultant Ricci Tensor and Ricci Scalar.

As an initial step, we need to calculate the contravariant metric. Knowing $g_{\mu\nu}$, we can compute the contravariant metric by solving $g^{\mu\nu}g_{\nu i} = \delta^{\mu}_{i}$, or by finding the inverse. However, if the original metric is diagonal, its inverse can be constructed from the individual inverses of the components. Let f_γ represent functions of time and space.

$$g_{\mu\nu} = \mathrm{diag}(f_0, f_1, f_2, f_3) \implies g^{\mu\nu} = \mathrm{diag}\left(\frac{1}{f_0}, \frac{1}{f_1}, \frac{1}{f_2}, \frac{1}{f_3}\right)$$

This means that $g_{\beta\beta} = f_\beta$ where f_β can be one of the four functions of time and space along the principle diagonal of the covariant metric.

Now we can move on to the Christoffel Symbols. Using equation (1) it is evident that for any given β, all components of $g^{\mu\nu}$ will vanish, except for when $\alpha = \beta$. We then have the following:

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\beta} \left\{ \frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right\} = \frac{1}{2 f_{\beta}} \left\{ \frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right\}$$

The above expression can be further broken down into five branches, depending on which of the values β, μ, ν are equivalent.

$$(a) \quad \Gamma^{\beta}_{\mu\nu} = \frac{1}{2 f_{\beta}} \left\{ \frac{\partial f_{\beta}}{\partial x^{\beta}} + \frac{\partial f_{\beta}}{\partial x^{\beta}} - \frac{\partial f_{\beta}}{\partial x^{\beta}} \right\} \quad \text{if} \quad \mu = \nu = \beta \tag{4}$$

$$(b) \quad \Gamma^{\beta}_{\mu\nu} = -\frac{1}{2 f_{\beta}} \frac{\partial f_{\mu}}{\partial x^{\beta}} \quad \text{if} \quad \mu = \nu \neq \beta$$

$$(c) \quad \Gamma^{\beta}_{\mu\nu} = \frac{1}{2 f_{\beta}} \frac{\partial f_{\beta}}{\partial x^{\mu}} \quad \text{if} \quad \mu \neq \nu = \beta \qquad (d) \quad \Gamma^{\beta}_{\mu\nu} = \frac{1}{2 f_{\beta}} \frac{\partial f_{\beta}}{\partial x^{\nu}} \quad \text{if} \quad \beta = \mu \neq \nu$$

$$(e) \quad \Gamma^{\beta}_{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu \neq \beta$$

2.2 $g_{11} = g_{22} = g_{33}$ Diagonal Metrics

$$g_{\mu\nu} = \begin{bmatrix} \ell(\vec{x}) & 0 & 0 & 0 \\ 0 & h(\vec{x}) & 0 & 0 \\ 0 & 0 & h(\vec{x}) & 0 \\ 0 & 0 & 0 & h(\vec{x}) \end{bmatrix}$$

If the metric is identical in its spacial diagonals, then we can solve for the Christoffel Symbols even more exactly. In the case where $g_{11} = g_{22} = g_{33} = h(\vec{x})$ and $g_{00} = \ell(\vec{x})$ we can split the computations of the Christoffel Symbols given in equation (4) into spacial and temporal parts. Considering only non zero components:

$$\Gamma^{0}_{00} = \frac{1}{2\ell} \left\{ \frac{\partial \ell}{\partial x^{0}} + \frac{\partial \ell}{\partial x^{0}} - \frac{\partial \ell}{\partial x^{0}} \right\} \implies \Gamma^{0}_{00} = \frac{1}{2\ell} \frac{\partial \ell}{\partial x^{0}}$$

$$\Gamma^{0}_{ii} = -\frac{1}{2h} \frac{\partial h}{\partial x^{0}}$$

$$\Gamma^{0}_{i0} = \frac{1}{2\ell} \frac{\partial \ell}{\partial x^{i}} \quad , \quad \Gamma^{0}_{0i} = \frac{1}{2\ell} \frac{\partial \ell}{\partial x^{i}} \quad \implies \quad \Gamma^{0}_{i0} = \Gamma^{0}_{0i} = \frac{1}{2\ell} \frac{\partial \ell}{\partial x^{i}}$$

$$\Gamma^{i}_{ii} = \frac{1}{2h} \left\{ \frac{\partial h}{\partial x^{i}} + \frac{\partial h}{\partial x^{i}} - \frac{\partial h}{\partial x^{i}} \right\} \implies \Gamma^{i}_{ii} = \frac{1}{2h} \frac{\partial h}{\partial x^{i}} \tag{5}$$

$$\Gamma^{i}_{\mu\mu} = -\frac{1}{2 f_{\mu}} \frac{\partial f_{\mu}}{\partial x^{i}} \quad \implies \quad \Gamma^{i}_{00} = -\frac{1}{2\ell} \frac{\partial \ell}{\partial x^{i}} \quad , \quad \Gamma^{i}_{jj} = -\frac{1}{2h} \frac{\partial h}{\partial x^{i}} \qquad i \neq j$$

$$\Gamma^i_{\mu i} = \frac{1}{2h}\frac{\partial h}{\partial x^\mu} \quad , \quad \Gamma^i_{i\nu} = \frac{1}{2h}\frac{\partial h}{\partial x^\nu} \qquad \Longrightarrow \qquad \Gamma^i_{0i} = \Gamma^i_{i0} = \frac{1}{2h}\frac{\partial h}{\partial x^0} \quad , \quad \Gamma^i_{ji} = \Gamma^i_{ij} = \frac{1}{2h}\frac{\partial h}{\partial x^j} \qquad i \neq j$$

Moving on to the Ricci Tensor, it is once again helpful to split the computation into spatial and temporal parts. Form equation (2) we arrive at:

$$\text{(case 1)} \quad R_{00} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{00} - \frac{\partial}{\partial x^0}\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha}$$

$$\text{(case 2)} \quad R_{0i} = R_{i0} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{0i} - \frac{\partial}{\partial x^i}\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{0i} - \Gamma^\alpha_{\beta i}\Gamma^\beta_{0\alpha} \tag{6}$$

$$\text{(case 3)} \quad R_{ii} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{ii} - \frac{\partial}{\partial x^i}\Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ii} - \Gamma^\alpha_{\beta i}\Gamma^\beta_{i\alpha}$$

$$\text{(case 4)} \quad R_{ij} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{ij} - \frac{\partial}{\partial x^j}\Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ij} - \Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} \qquad i \neq j$$

2.3 Aside: Python Expression Solver

Computing these components of the Ricci Tensor is rather contrived, as thus it may be useful to develop a technique with with to solve these calculations computationally.

I developed a python notebook that could do these calculations. You can find it here on my GitHub: **https://github.com/SorenHelhoski/LHS-Einstein-Solver** I accomplished this by utilizing *list* object editing, and by converting Laurent series strings into multidimensional arrays. For example, I would express the polynomial $1 + 2x + xy^2$ as the array $[\ [1, [\ ], [\ ]],\ [2, ['x'], [1]],\ [1, ['x', 'y'], [1, 2]]\ ]$. Each element of the array is a term in the polynomial represented as another array. The first term in this array is the coefficient, the second list contains the names of all the variables in the term, and the final list contains the respective powers of the variables.

Once I had decomposed any expression into this form, I could add, multiply, and partial differentiate any expression by adding and multiplying the powers and the coefficients in certain ways. The only difficulty was implementing functions of other variables. Functions add and multiply like variables, but require extra information to partial differentiate. I added in a system of defining the chain rules of any given function to solve this issue.

2.4 Computing the Ricci Tensor

Using the software described above, we can compute the solutions to equations (6). For this, we adapt a new notation in which $f_{,\mu} = \frac{\partial f}{\partial x^\mu}$ Recall that the metric is $\text{diag}(\ell(\vec{x}), h(\vec{x}), h(\vec{x}), h(\vec{x}))$

Tradiational summation notation is adopted here in order to clarify that the summations over k are performed only once — even though some terms are indexed by k twice.

$$R_{00} = -\frac{3h_{,00}}{2h} + \frac{3(h_{,0})^2}{4(h)^2} + \frac{3\ell_{,0}h_{,0}}{4\ell h} + \sum_k \left(-\frac{\ell_{,kk}}{2h} - \frac{\ell_{,k}h_{,k}}{4(h)^2} + \frac{(\ell_{,k})^2}{4\ell h} \right)$$

$$R_{0i} = R_{i0} = \frac{h_{,0}h_{,i}}{(h)^2} - \frac{h_{,0i}}{h} + \frac{\ell_{,i}h_{,0}}{2\ell h} \tag{7}$$

$$R_{ii} = \frac{\ell_{,0}h_{,0}}{4(\ell)^2} - \frac{h_{,00}}{2\ell} - \frac{(h_{,0})^2}{4\ell h} + \frac{1}{4}\left(\frac{\ell_{,i}}{\ell}\right)^2 - \frac{1}{2}\frac{\ell_{,ii}}{\ell} + \frac{3}{4}\left(\frac{h_{,i}}{h}\right)^2 - \frac{1}{2}\frac{h_{,ii}}{h} + \frac{1}{2}\frac{\ell_{,i}h_{,i}}{\ell h} + \sum_k \left(\frac{1}{4}\left(\frac{h_{,k}}{h}\right)^2 - \frac{1}{2}\frac{h_{,kk}}{h} - \frac{1}{4}\frac{\ell_{,k}h_{,k}}{\ell h} \right)$$

$$R_{ij} = R_{ji} = \frac{3h_{,i}h_{,j}}{4(h)^2} - \frac{h_{,ij}}{2h} + \frac{\ell_{,i}\ell_{,j}}{4(\ell)^2} - \frac{\ell_{,ij}}{2\ell} + \frac{\ell_{,i}h_{,j}}{4\ell h} + \frac{\ell_{,j}h_{,i}}{4\ell h} \qquad i \neq j$$

2.5 Computing the Ricci Scalar

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} = g^{0\nu}R_{0\nu} + g^{i\nu}R_{i\nu}$$

Computing the Ricci Scalar is done by contracting the Ricci Tensor with the contravariant metric. Since we are only dealing with diagonal metrics, this contraction can be expressed as a simpler sum.

$$\mathcal{R} = g^{00}R_{00} + \sum_i g^{ii}R_{ii}$$

:

Recalling that $g^{00} = 1/\ell(\vec{x})$ and $g^{ii} = 1/h(\vec{x})$

$$g^{00}R_{00} = -\frac{3h_{,00}}{2\ell h} + \frac{3(h_{,0})^2}{4\ell(h)^2} + \frac{3\ell_{,0}h_{,0}}{4(\ell)^2 h} + \sum_k \left(-\frac{\ell_{,kk}}{2\ell h} - \frac{\ell_{,k}h_{,k}}{4\ell(h)^2} + \frac{(\ell_{,k})^2}{4(\ell)^2 h} \right)$$

$$\sum_i g^{ii}R_{ii} = \frac{1}{h}\sum_i R_{ii}$$

Adding up the two expressions above will yield the Ricci Scalar for a spatially identical diagonal matrix. When simplified, the Ricci Scalar is:

$$\mathcal{R} = -\frac{3h_{,00}}{\ell h} + \frac{3\ell_{,0}h_{,0}}{2(\ell)^2 h} + \sum_k \left(-\frac{\ell_{,kk}}{\ell h} - \frac{\ell_{,k}h_{,k}}{2\ell(h)^2} + \frac{(\ell_{,k})^2}{2(\ell)^2 h} + \frac{3(h_{,k})^2}{2(h)^3} - \frac{2h_{,kk}}{(h)^2} \right) \tag{8}$$

2.6 Sanity Check with the Flat Metric

We can test the above formulas to find the Ricci Tensor and Ricci Scalar in the case of the flat metric. In this specific case, the functions $h(\vec{x})$ and $\ell(\vec{x})$ are both constant. Since equations (7) and (8) only contain terms which have numerators that are composed of derivatives of ℓ and h, both the Ricci Tensor and Ricci Scalar will be zero.

3 The Friedman Robertson Walker Metric

We can now apply this method to the Friedman Robertson Walker metric. This metric describes a homogeneous, isotropic, expanding, path connected universe.

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)) \implies g_{\mu\nu} = \text{diag}(-1, a^{-2}(t), a^{-2}(t), a^{-2}(t))$$

Right away we can compute the Ricci Tensor and Scalar with equations (7) and (8). To make things simpler, ℓ is a constant, and h is only a function of time. This means that we can immediately eliminate all terms that contain any derivative of ℓ, or a spatial derivative of h. Also note that $h = a^2 \implies h_{,00} = 2(a\ddot{a} + \dot{a}^2)$

$$R_{00} = -\frac{3h_{,00}}{2h} + \frac{3(h_{,0})^2}{4(h)^2} = -\frac{3\ddot{a}}{a}$$

$$R_{0i} = R_{i0} = 0$$

$$R_{ii} = -\frac{h_{,00}}{2\ell} - \frac{(h_{,0})^2}{4\ell h} = a\ddot{a} + 2\dot{a}^2$$

$$R_{ij} = R_{ji} = 0$$

$$\mathcal{R} = -\frac{3h_{,00}}{\ell h} = \frac{6\ddot{a}}{a} + \frac{6\dot{a}^2}{a^2}$$

4 The Perturbed Metric

Starting with the Friedman Robertson Walker metric again, we can now add slight temporal and spatial perturbations, called Ψ and Φ respectfully.

$$g_{\mu\nu} = \text{diag}(-1 - 2\Psi, a^2(1 + 2\Phi), a^2(1 + 2\Phi), a^2(1 + 2\Phi))$$

Meaning that the covariant metric is:

$$g^{\mu\nu} = \text{diag}\left(\frac{1}{-1 - 2\Psi}, \frac{1}{a^2(1 + 2\Phi)}, \frac{1}{a^2(1 + 2\Phi)}, \frac{1}{a^2(1 + 2\Phi)}\right)$$

Under the assumption that the spacial and temporal perturbations are small, the rational functions given by the covariant metric above can be rewritten in first order. This approximation can also be applied to the derivatives of Φ and Ψ.

$$g^{\mu\nu} = \text{diag}(-1 + 2\Psi, a^2(1 - 2\Phi), a^2(1 - 2\Phi), a^2(1 - 2\Phi))$$

The method we used to derive equations (7) and (8) will not be completely altered by this change. The components of the contravariant metric only appear in the denominators of every term, meaning that we can easily identify when to use components of the covariant versus the contravariant metric.

However, making a first order approximation at this stage of the computation will require us to continue considering only first order terms.

$$\frac{\partial}{\partial x^\mu}\ell(\vec{x}) = \frac{\partial}{\partial x^\mu}(-1+2\Psi) = 2\frac{\partial}{\partial x^\mu}\Psi \qquad \frac{\partial}{\partial x^i}h(\vec{x}) = \frac{\partial}{\partial x^i}(a^2 - 2a^2\Phi) = 2a^2\frac{\partial}{\partial x^i}\Phi$$

$$\frac{\partial}{\partial x^0}h(\vec{x}) = \frac{\partial}{\partial x^0}(a^2 - 2a^2\Phi) = 2a\dot{a} + \mathcal{O}(1) \quad \text{Let } \bar{h} = 2a\dot{a}$$

First derivatives of ℓ will be first order, and first spacial derivatives of h will also be first order. Additionally, the time derivative of h is first order, except for one zeroth order term. Keeping this in mind, we can simplify equations (7) and (8) slighlty, by eliminating products of the above derivatives. We begin by only considering the numerators of all the terms.

$$R_{00} = -\frac{3h_{,00}}{2h} + \frac{3\bar{h}^2}{4(h)^2} + \frac{3\ell_{,0}\bar{h}}{4\ell h} + \sum_k \left(-\frac{\ell_{,kk}}{2h}\right)$$

$$R_{0i} = R_{i0} = \frac{\bar{h}h_{,i}}{(h)^2} - \frac{h_{,0i}}{h} + \frac{\ell_{,i}\bar{h}}{2\ell h} \tag{9}$$

$$R_{ii} = \frac{\ell_{,0}\bar{h}}{4(\ell)^2} - \frac{h_{,00}}{2\ell} - \frac{(h_{,0})^2}{4\ell h} - \frac{1}{2}\frac{\ell_{,ii}}{\ell} - \frac{1}{2}\frac{h_{,ii}}{h} + \sum_k \left(-\frac{1}{2}\frac{h_{,kk}}{h}\right)$$

$$R_{ij} = R_{ji} = -\frac{h_{,ij}}{2h} + \frac{\ell_{,i}\ell_{,j}}{4(\ell)^2} - \frac{\ell_{,ij}}{2\ell} \qquad i \neq j$$

$$\mathcal{R} = -\frac{3h_{,00}}{\ell h} + \frac{3\ell_{,0}\bar{h}}{2(\ell)^2 h} + \sum_k \left(-\frac{\ell_{,kk}}{\ell h} - \frac{2h_{,kk}}{(h)^2}\right) \tag{10}$$

At this point, the numerators are all at least first order terms. This means that if we exchange out $1/\ell$ and $1/h$ for the respective terms in the contravariant metric, we only need to multiply by the zeroth order terms for every $1/\ell$ and $1/h$.

$$R_{00} = -\frac{3}{2}h_{,00}a^2 + \frac{3}{4}\bar{h}^2a^4 - \frac{3}{4}\ell_{,0}\bar{h}a^2 + \sum_k \left(-\frac{1}{2}\ell_{,kk}a^2\right)$$

$$R_{0i} = R_{i0} = \bar{h}h_{,i}a^4 - h_{,0i}a^2 - \ell_{,i}\bar{h}a^2 \tag{11}$$

$$R_{ii} = \frac{1}{4}\ell_{,0}\bar{h} + \frac{1}{2}h_{,00} + \frac{1}{4}(h_{,0})^2a^2 + \frac{1}{2}\ell_{,ii} - \frac{1}{2}h_{,ii}a^2 + \sum_k \left(-\frac{1}{2}h_{,kk}a^2\right)$$

$$R_{ij} = R_{ji} = -\frac{1}{2}h_{,ij}a^2 + \frac{1}{4}\ell_{,i}\ell_{,j} + \frac{1}{2}\ell_{,ij} \qquad i \neq j$$

$$\mathcal{R} = 3h_{,00}a^2 + \frac{3}{2}\ell_{,0}\bar{h}a^2 + \sum_k \left(\ell_{,kk}a^2 - 2h_{,kk}a^4\right) \tag{12}$$

We can make one final simplification for this metric, that being the following. When i is not an index it represents the imaginary number

$$\frac{\partial\Psi}{\partial x^i} = ik_i\Psi \qquad \frac{\partial\Phi}{\partial x^i} = ik_i\Phi \qquad H = \frac{\dot{a}}{a}$$

Using these simplifications, we could plug the results back into the equations given in (7). Most of these expressions will be very large, so for sake of space, I'll only include the R_{00} term below.

$$-1\Psi a^{-2}k_x^2 - 1\Psi a^{-2}k_y^2 - 1\Psi a^{-2}k_z^2 - 3a^{-1}\ddot{a} - 3\Phi_{,00} + 3\Psi_{,0}a^{-1}\dot{a} - 6\Phi_{,0}a^{-1}\dot{a}$$

Plugging in and simplifying equation (12); the final form of the Ricci Scalar for the Perturbed Friedman Robertson Walker metric can be expressed as the following.

$$\mathcal{R} = 2\Psi a^{-2}k_1^2 + 2\Psi a^{-2}k_2^2 + 2\Psi a^{-2}k_3^2 + 6a^{-1}\ddot{a} + 6\Phi_{,00} - 6\Psi_{,0}a^{-1}\dot{a} + 24\Phi_{,0}a^{-1}\dot{a} +$$

$$-12\Psi a^{-1}\ddot{a} + 6H^2 - 12\Psi H^2 + 4\Phi a^{-2}k_1^2 + 4\Phi a^{-2}k_2^2 + 4\Phi a^{-2}k_3^2$$

5 Remarks on the Method

It is clear that for more complicated covariant metrics, the computations required to achieve the Ricci Scalar and Ricci Tensor become increasingly complex. This reaches a point where deriving the Einsteins equations by hand is almost impossible to do in a reasonable amount of time. Fortunately, most metrics will at least have some symmetries that greatly simplify the process.

In most well-known cases, the covariant metric can be diagonalized, and thus the equations given in (4) can easily be used to calculate the Christoffel symbols.

Further, if the spatial components of a diagonal metric are identical, then we can go a step further and use equation (7) and (8) directly to calculate Ricci Scalars and Ricci Tensors. Unfortunately, this simplification is a result the coordinate system, thus, some symmetrical metrics will not be able to use the formulas derived.