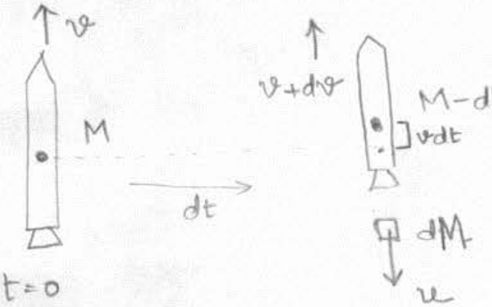
If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

System of varying mass, Rocket propulsion

The rocket emits fuel with constant relative velocity wrt rocket
Let the velocity of fuel emitted by u wrt inertial frame

Rocket of mass M
initial velocity v



Note: The CM continues to move with the same initial velocity v but the rocket (of new mass $M-dm$) moves with velocity $(v+dv)$ after dt time

Conservation of linear momentum,

$$P_i = P_f \Rightarrow Mv = (M-dM)(v+dv) + dM(-u)$$

$$\Rightarrow \cancel{Mv} = \cancel{Mv} + M\,dv - v\,dM - \underbrace{dM\,dv}_{\text{too small neglect.}} - u\,dM$$

$$\Rightarrow u\,dm = M\,dv - v\,dM$$

$$+(u+v)\,dM = M\,dv \Rightarrow dv = \frac{1}{M}(u+v)\,dM$$

$$\int_{v_i}^{v_f} dv = +\underbrace{(v+u)}_{-(v+u)\,=\,\text{velocity of rocket w.r.t fuel} \,=\, v_{rel} \,=\, \text{constant as given}} \int_{M_i}^{M_f} \frac{dM}{M} \Rightarrow v_f - v_i = -v_{rel} \ln\left(\frac{M_f}{M_i}\right)$$

$$\Rightarrow \boxed{v_f - v_i = v_{rel} \ln\left(\frac{M_i}{M_f}\right)}$$

Average acceleration,

$$\boxed{a_{avg} = \frac{v_f - v_i}{t_f - t_i} = \frac{v_{rel}}{t_f - t_i} \ln\left(\frac{M_i}{M_f}\right)}$$

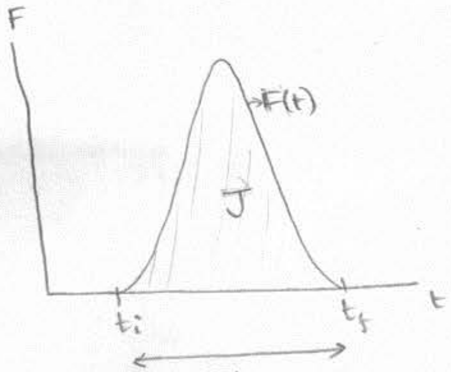
Collisions

* Impulse of a force :

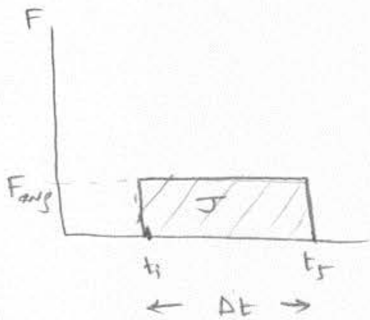
Newtons 2^{nd} law, $\quad \dfrac{d\vec{p}}{dt} = \vec{F}(t)$

$$\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}\, dt \quad \Rightarrow \quad \vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\, dt$$

Impulse of a force



Δt is a short time interval



$$\vec{P}_f - \vec{P}_i = \Delta\vec{P} = \int_{t_i}^{t_f} \vec{F}\, dt = \vec{J}$$

$\Delta\vec{P} = 0$ if there is no net force

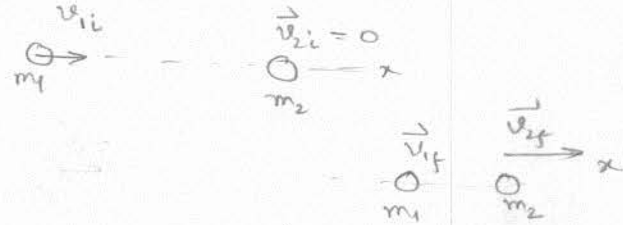
* Kinetic Energy Conservation
* Linear Momentum Conservation

] Important ideas for studying collisions.

Collisions

- → * Elastic collision [KE & linear momentum conserved]
- → Partially inelastic collision [linear momentum conserved
  KE not conserved
  bodies don't stick together, but are deformed]
- → * Completely inelastic collision [linear mom. is conserved
  KE is not conserved
  bodies stick together & deformed]

Elastic Collision



Total $KE_i = KE_f$ Total

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

linear momentum conservation

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\Rightarrow m_1 (v_{1i} - v_{1f}) = m_2 v_{2f}$$

$$\rightarrow m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2 v_{2f}^2 .$$

$$\Rightarrow (v_{1i} + v_{1f}) = v_{2f} \xrightarrow[m_1]{\text{mult by}} m_1 (v_{1i} + v_{1f}) = m_1 v_{2f}$$

→ Solve for v_{1f} & v_{2f}

$$\boxed{v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}}$$

→ Check what happens

(i) if $m_1 = m_2$

$v_{1f} = 0$ & $v_{2f} = v_{1i}$

velocities exchanged after collision

(ii) Massive target $m_2 \gg m_1$

$v_{1f} \approx - v_{1i}$ $\qquad v_{2f} \approx \underbrace{\left(\frac{2m_1}{m_2}\right)}_{\text{very less}} v_{1i}$

eg: bouncing of a ball elastically against a wall

(iii) $m_1 \gg m_2$ Then $\underbrace{v_{1f} \approx v_{1i}}_{\text{continues moving}}$ and $\underbrace{V_{2f} \approx 2V_{1i}}$ → twice the speed.

eg: cannon ball & golf ball shot at each other

Moving targets ($v_{2i} \neq 0$)

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

$$m_1 (v_{1i} - v_{1f}) = -m_2 (v_{2i} - v_{2f})$$

$$m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) = -m_2 (v_{2i} - v_{2f})(v_{2i} + v_{2f})$$

divide and work out

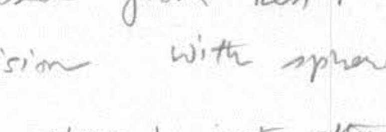
$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

example: Two metal spheres, suspended by vertical cords, initially just touch, as in the figure. Sphere 1 with mass $m_1 = 30g$, is pulled to the left to height $h_1 = 8.0$ cm and then released from rest. After swinging down it undergoes an elastic collision with sphere 2, whose mass $m_2 = 75$ g. What is the velocity v_{1f} of sphere 1 just after the collision? What height does the sphere 2 reach after collision?

$$\frac{1}{2} m_1 v_{1i}^2 = m_1 g h_1$$

$$v_{1i} = \sqrt{2gh} = \sqrt{2 \; 9.8 \; 0.080} = 1.252 \text{ m/s}$$

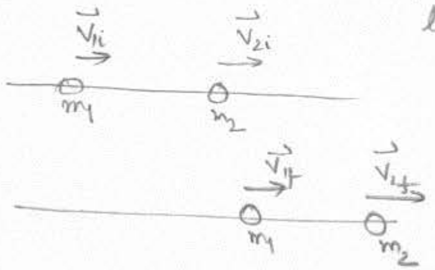


$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{0.030 - 0.075}{0.030 + 0.075} (1.252 \text{ m/s}) = -0.537 \text{ m/s}$$

$$\approx -0.54 \text{ m/s}$$

The minus sign tells us that sphere 1 moves left just after collision.

Inelastic collision (partially)

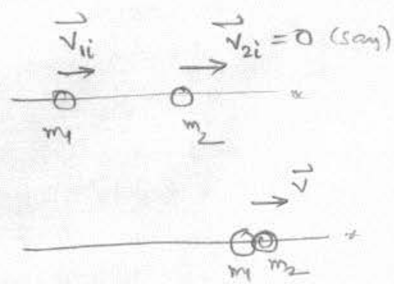


linear Momentum is conserved

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

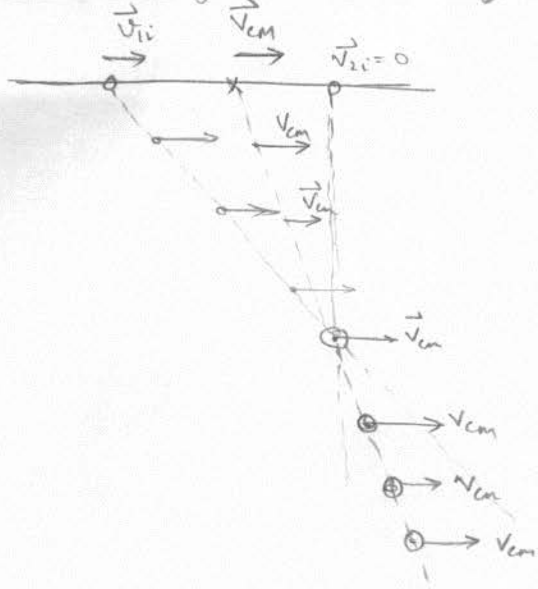
Completely inelastic collision



$$m_1 v_{1i} = (m_1 + m_2) V$$

$$V = \frac{m_1}{m_1 + m_2} v_{1i}$$

Velocity of the center of mass



$$\vec{P}_{CM\,i} = \vec{P}_{1i} + \vec{P}_{2i}$$

$$\vec{P}_{CM\,f} = M\vec{v}_{CM} = (m_1 + m_2)\vec{v}_{CM} \left[= (m_1 + m_2)\vec{V} \right]$$

final velocity is the velocity of CM and it doesnot change from initial case

$$\vec{P}_{cm\,f} = M\vec{v}_{CM} = \vec{P}_{cm\,i} = \vec{P}_{1i} + \vec{P}_{2i}$$

$$\boxed{\vec{v}_{cm} = \frac{\vec{P}_{1i} + \vec{P}_{2i}}{(m_1 + m_2)}}$$

A ballastic pendulum consists of a large block of wood of mass $M = 5.4$ kg, hangind from two long cords. A bullet of mass $m = 9.5$ g is fired into the block, coming quickly to rest.

The block + bullet then swing upward, their CM rise a vertical distance $h = 6.3$ cm before coming to momentary rest.

What is the speed of the bullet just prior to collision?

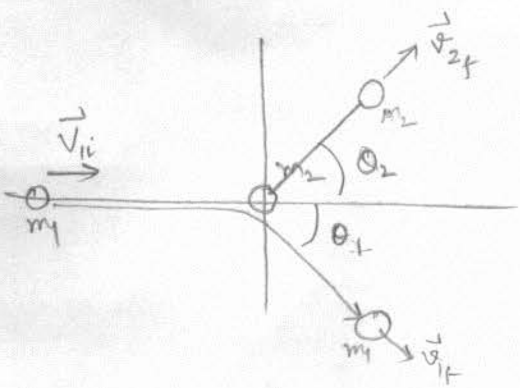
$$(m+M)V = mv \quad \Rightarrow \quad V = \frac{mv}{m+M} \quad \text{find } v?$$

$$\frac{1}{2}(m+M)V^2 = (m+M)gh$$

$$v = \frac{m+M}{m}\sqrt{2gh} = \left(\frac{0.0095\text{ kg} + 5.4\text{ kg}}{0.0095\text{ kg}}\right)\sqrt{2(9.8)(0.063)}\text{ m}$$

$$v = 630\text{ m/s}$$

Collisions in 2 dimensions



$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}$$

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

$$\left.\begin{aligned} m_1 v_{1i} &= m_1 v_{1f}\cos\theta_1 + m_2 v_{2f}\cos\theta_2 \\ 0 &= -m_1 v_{1f}\sin\theta_1 + m_2 v_{2f}\sin\theta_2 \end{aligned}\right\}$$

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \quad (KE)$$

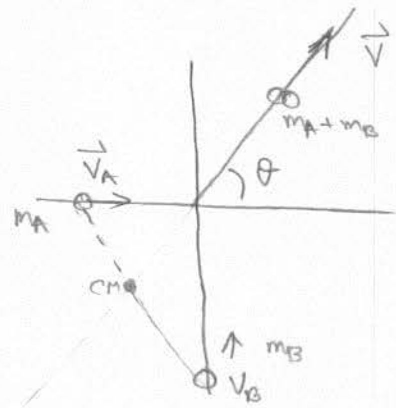
3 equations, $\underbrace{7 \text{ variables}}$ (if 4 are known) then the rest (3) can be solved for,

$m_1 m_2$, v_{1i} v_{2f}, v_{1f}, θ_1, θ_2

Two skaters collide and catch each other (completely inelastic collision). Place the origin at the point of collision

a) What is the velocity after collision

b) What is the $\vec{v}_{CM}$ before and after the collision.



a) $m_A \vec{v}_A + m_B \vec{v}_B = (m_A + m_B)\vec{V}$

$$\vec{V} = \frac{m_A \vec{v}_A + m_B \vec{v}_B}{m_A + m_B}$$

X axis:

$$m_A v_A + m_B 0 = (m_A + m_B) V \cos\theta$$

Y axis

$$m_A 0 + m_B v_B = (m_A + m_B) V \sin\theta$$

$$\tan\theta = \frac{m_B v_B}{m_A v_A} \Rightarrow \theta = \tan^{-1}\left(\frac{m_B v_B}{m_A v_A}\right)$$

$$V = \frac{m_B v_B}{(m_A + m_B)\sin\theta}$$

b) $\vec{v}_{CM} = \vec{V}$