The component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

**System of varying mass, Rocket propulsion**

Rocket of mass $M$

initial velocity $v$

$\uparrow v$

$\downarrow u$

$t=0$

$\frac{dv}{dt}$

$\Delta M$

$\Delta t$

$M - \Delta M$

$M - \Delta M$

$M - \Delta M$

Note: The CM continues to move with the same initial velocity but the rocket (of new mass $M - \Delta M$) moves with velocity $(v + \Delta v)$ after time.

Conservation of linear momentum,

$$p_i = p_f \implies Mv = (M - \Delta M)(v + \Delta v) + \Delta M(-u)$$

$$\implies M\Delta v = M\Delta v - u\Delta M - \Delta Mdv - u\Delta M$$

$$\implies u\Delta M = M\Delta v = \Delta M + (u + v)\Delta M$$

$$\implies dv = \frac{1}{M} (u + v) \Delta M$$

$$\int dv = \left[ \frac{v}{v_i} \right] \Rightarrow v_f - v_i = -\frac{v_i}{v_f} ln \left( \frac{M_f}{M_i} \right)$$

$$(v_f - v_i) = \text{velocity of rocket wrt fuel} = v_i = \text{constant as given}$$

$$(v_f - v_i) = \text{velocity of rocket wrt fuel} = v_i$$

$$(v_f - v_i) = \text{constant as given}$$
Average acceleration,

$$a_{ave} = \frac{v_f - v_i}{t_f - t_i} = \frac{v_{rel}}{t_f - t_i} \ln \left( \frac{M_i}{M_f} \right)$$

**Collisions**

- **Impulse of a force:**

  Newton 2nd law,
  
  $$\frac{d\vec{p}}{dt} = \vec{F}(t)$$

  $$\int_{p_i}^{p_f} dp = \int_{t_i}^{t_f} F \, dt \Rightarrow \quad \vec{J} = \int_{t_i}^{t_f} \vec{F}(t) \, dt$$

  Impulse of a force

  \[ \text{Dt is a short time interval} \]

  \[ \vec{p}_f - \vec{p}_i = \Delta \vec{p} = \int_{t_i}^{t_f} F \, dt = \frac{\vec{J}}{t_f - t_i} \]

  \[ \Delta \vec{p} = 0 \text{ if } \text{there is no force} \]

- **Kinetic Energy Conservation**
- **Linear Momentum Conservation**

**Important ideas for studying collisions.**
Collisions

- Elastic collision [KE & linear momentum conserved]
- Partially inelastic collision [linear momentum conserved, KE not conserved, bodies don't stick together, but one deformed]
- Completely inelastic collision [linear mom is conserved, KE is not conserved, bodies stick together & deformed]

Elastic Collision

\[
\text{Total KE}_i = \text{KE}_f \text{ Total}
\]
\[
\frac{1}{2} m_1 v_{i1}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2
\]

Linear momentum conservation

\[
m_1 v_{i1} = m_1 v_{1f} + m_2 v_{2f}
\]
\[
\Rightarrow m_1 (v_{i1} - v_{1f}) = m_2 v_{2f}
\]

\[
m_1 (v_{i1} - v_{1f}) (v_{i1} + v_{1f}) = m_2 v_{2f}^2
\]
\[
\Rightarrow (v_{i1} + v_{1f}) = v_{2f} \frac{m_1}{m_1} (v_{i1} + v_{1f}) = m_1 \frac{v_{2f}}{m_1}
\]

Check what happens

(i) if \(m_1 = m_2\)

\[
v_{1f} = 0 \quad \text{and} \quad v_{2f} = v_{i1}
\]

velocities exchanged after collision

(ii) Massive target \(m_2 \gg m_1\)

\[
v_{1f} \approx -v_{i1} \quad v_{2f} \approx \left(\frac{2m_1}{m_2}\right) v_{i1}
\]

\(\downarrow\) very ten

eg: bouncing of a ball elastically against a wall
(iii) \( m_1 \gg m_2 \)  

Then \( V_{1f} \approx V_{1i} \) and \( V_{2f} \approx 2V_{1i} \)

\text{e.g.: cannon ball a golf ball shot at each other \( \approx \) twice the speed.}

\underline{Moving Targets} \quad (V_{2i} \neq 0)

\[
egin{align*}
    m_1v_{1i} + m_2v_{2i} &= m_1v_{1f} + m_2v_{2f} \\
    \frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 &= \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 \\
    m_1(v_{1i} - v_{1f}) &= -m_2(v_{2i} - v_{2f}) \\
    m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) &= -m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f})
\end{align*}
\]

divide and work out

\[
egin{align*}
    v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \\
    v_{2f} &= \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}
\end{align*}
\]

\text{example: Two metal spheres, suspended by vertical cords, initially just touch, as in the figure. Sphere 1 with mass \( m_1 = 30g \), is pulled to the left to height \( h = 8.0 \text{ cm} \) and then released from rest. After swinging down it undergoes an elastic collision with sphere 2, whose mass \( m_2 = 75g \). What is the velocity \( V_{1f} \) of sphere 1 just after the collision? What height does the sphere 2 reach after collision?}

\[
\frac{1}{2}m_1v_{1i}^2 = m_1gh
\]

\[
V_{1i} = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 0.080} = 1.252 \text{ m/s}
\]

\[
V_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{0.030 - 0.075}{0.030 + 0.075} (1.252 \text{ m/s}) = -0.537 \text{ m/s}
\]

\[z = -0.54 \text{ m/s}\]

The minus sign tells us that sphere 1 moves left just after collision.
Inelastic collision (partially)

Linear momentum is conserved

\[ \mathbf{P}_{1i} + \mathbf{P}_{2i} = \mathbf{P}_{1f} + \mathbf{P}_{2f} \]

\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \]

Completely inelastic collision

\[ m_1 v_{1i} = (m_1 + m_2) V \]

\[ V = \frac{m_1}{m_1 + m_2} v_{1i} \]

Velocity of the center of mass

\[ \mathbf{P}_{CM} i = \mathbf{P}_{1i} + \mathbf{P}_{2i} \]

\[ \mathbf{P}_{CM f} = M \mathbf{v}_{CM} = (m_1 + m_2) \mathbf{v}_{CM} \quad [= (m_1 + m_2) \mathbf{V}] \]

Final velocity is the velocity of CM and it does not change from initial case

\[ \mathbf{P}_{CM f} = M \mathbf{v}_{CM} = \frac{\mathbf{P}_{1i} + \mathbf{P}_{2i}}{m_1 + m_2} \]

\[ \mathbf{v}_{CM} = \frac{\mathbf{P}_{1i} + \mathbf{P}_{2i}}{m_1 + m_2} \]
A ballistic pendulum consists of a large block of wood of mass \( M = 5.4 \text{ kg} \) hanging from two long cords. A bullet of mass \( m = 9.5 \text{ g} \) is fired into the block, coming to rest.

The block-bullet then swings upward, their CM rise = vertical distance \( h = 0.3 \text{ cm} \) before coming to momentarily rest.

What is the speed of the bullet just prior to collision?

\[
(m+M)\dot{v} = m\dot{v} \quad \Rightarrow \quad \dot{v} = \frac{m\dot{v}}{m+M}
\]

\[
\frac{1}{2} (m+M) \dot{v}^2 = (m+M)gh
\]

\[
\dot{v} = \frac{m+M}{m} \sqrt{2gh} = \frac{(0.0095 \text{ kg} + 5.4 \text{ kg})}{0.0095 \text{ kg}} \sqrt{2(9.8)(0.030)}
\]

\[
\dot{v} = 630 \text{ m/s}
\]

**Collisions in 2 dimensions**

\[
\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_f + \vec{P}_{2f}
\]

\[
k_{1i} + k_{2i} = k_{1f} + k_{2f}
\]

\[
m_1v_{1i} = m_1v_{1f} \cos \theta_1 + m_2v_{2f} \cos \theta_2
\]

\[
c_0 = -m_1v_{1f} \sin \theta_1 + m_2v_{2f} \sin \theta_2
\]

\[
\frac{1}{2} m_1v_{1i}^2 = \frac{1}{2} m_1v_{1f}^2 + \frac{1}{2} m_2v_{2f}^2 \quad (KE)
\]

3 equations, 7 variables (5 are unknown) then the rest (3) can be solved for, \( m, m_2, v_{1i}, v_{1f}, v_{2f}, \theta_1, \theta_2 \)
Two skaters collide and catch each other (completely inelastic collision). Place the origin at the point of collision.

a) What is the velocity after collision?

b) What is the $\vec{V}_{cm}$ before and after the collision.

d) $\vec{V}_{cm} = \vec{V}$

\[ m_A \vec{v}_A + m_B \vec{v}_B = (m_A + m_B) \vec{V} \]

\[ \vec{V} = \frac{m_A \vec{v}_A + m_B \vec{v}_B}{m_A + m_B} \]

X axis:
\[ m_A \vec{v}_A + m_B \vec{v}_B = (m_A + m_B) \vec{V} \cos \theta \]

Y axis:
\[ m_A \vec{v}_A + m_B \vec{v}_B = (m_A + m_B) \vec{V} \sin \theta \]

\[ \tan \theta = \frac{m_B \vec{v}_B}{m_A \vec{v}_A} \quad \Rightarrow \quad \theta = \tan^{-1} \left( \frac{m_B \vec{v}_B}{m_A \vec{v}_A} \right) \]

\[ \vec{V} = \frac{m_B \vec{v}_B}{(m_A + m_B) \sin \theta} \]