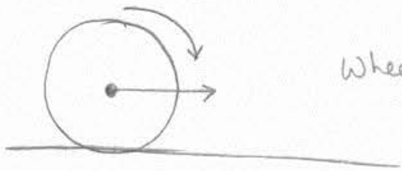
Rotation.

Any motion can be considered as a superposition of pure rotation and pure translation

example:

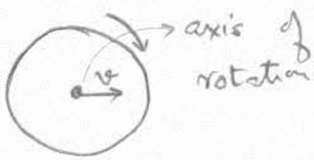


Wheel rolling on a road.

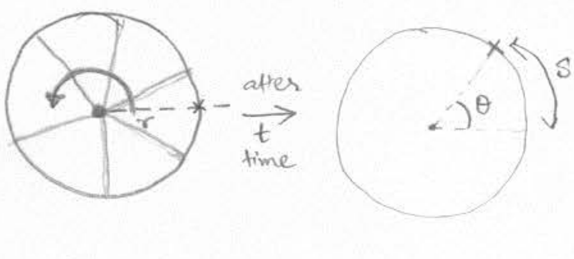
Rigid body is a body that can rotate with all its parts locked together and without any change in its shape.

Rotation occurs about an axis.

the axis can be moving with a velocity



Consider a wheel rotating about a fixed axis



$\theta = \frac{s}{r}$ (measured in radians)

θ is the angular position

Angular displacement, $\Delta\theta = \theta_2 - \theta_1$

Angular displacement in counterclockwise direction is positive, and in clockwise direction is negative.

[units of $\Delta\theta$ is rad]

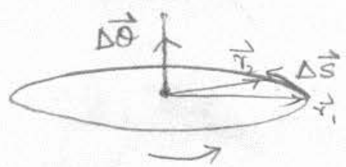
Angular velocity

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

instantaneous angular velocity, $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$

The magnitude of angular velocity is angular speed, which is also represented by ω. [units is $rad.sec^{-1}$]

Direction of angular velocity.



$\vec{r}_2 - \vec{r}_1 = \Delta\vec{s}$

$$\boxed{\Delta\vec{\theta} \times \vec{r} = \Delta\vec{s}}$$

* Only very small angular quantities can be treated as vectors.

$\Delta\vec{\theta}$ & $\vec{r}$ are always $\perp$ to each other [$\therefore \sin\frac{\pi}{2} = 1$]

Taking magnitudes, $\Delta\theta\, r = \Delta s \Rightarrow \boxed{\Delta\theta = \frac{\Delta s}{r}}$

$\therefore$ The direction of $\Delta\vec{\theta}$ is along the axis of rotation and it is got by using the right hand rule, swirl your fingers in the direction of $\Delta\vec{s}$ (displacement) / direction of rotation and the thumb points at the $\Delta\vec{\theta}$.

Now, $\vec{\omega} = \lim_{\Delta t \to 0} \left(\frac{\Delta\vec{\theta}}{\Delta t}\right)$ (in the same direction as $\Delta\vec{\theta}$) for a uniform motion with constant $\vec{\omega}$

$$\vec{\omega} = \frac{d\vec{\theta}}{dt}$$

Angular acceleration

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad [\text{unit: } rad\ sec^{-2}]$$

Angular acceleration is zero for a uniform circular motion.

Rotation with constant Angular Acceleration : α

$$v = v_0 + at \longrightarrow \omega = \omega_0 + \alpha t$$

$$(x - x_0) = v_0 t + \frac{1}{2}at^2 \longrightarrow (\theta - \theta_0) = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$v^2 = v_0^2 + 2a(x - x_0) \longrightarrow \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Relating the linear & angular variables

$$\theta = \frac{s}{r} \Rightarrow \boxed{s = \theta r} \quad (i)$$

$$v = \frac{ds}{dt} = \frac{d(\theta r)}{dt} = \left(\frac{d\theta}{dt}\right)r + \theta\left(\frac{dr}{dt}\right)$$

$$= \omega r + \theta \cancel{\left(\frac{dr}{dt}\right)}$$

if circular motion then r doesnot change

$$\frac{dr}{dt} = 0$$

for circular motion, $\boxed{v = \omega r}$ - (ii)

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

$$a = \frac{dv}{dt} = \frac{d(\omega r)}{dt} = \left(\frac{d\omega}{dt}\right)r + \omega\cancel{\left(\frac{dr}{dt}\right)}$$

0 circular motion

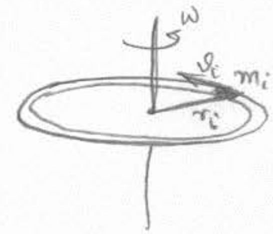
$$a = \frac{dv}{dt} = \alpha r \text{ (for circular motion)}$$

$$\therefore \boxed{a_t = \alpha r \quad \text{and} \quad a_r = \frac{v^2}{r} = \omega^2 r}$$

Kinetic energy of Rotation.

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \dots$$
$$= \sum \frac{1}{2} m_i v_i^2$$

example



$v_i = \omega r_i$

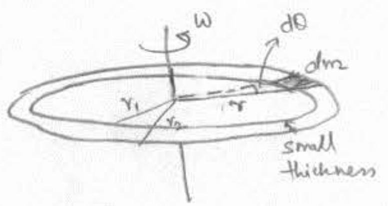
$$K = \sum \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

define $I = \sum m_i r_i^2$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

Calculating K.E for continuous mass.

Total mass is M



$$dK = \frac{1}{2}(dm)v^2 = \frac{1}{2}(dm)(\omega r)^2$$

$$dm = (r\, d\theta)\rho$$

ρ = linear density

$$dK = \frac{1}{2}\, \underbrace{r\, d\theta}\; \rho\, (\omega r)^2$$

$$\rho = \frac{M}{(2\pi r)}$$

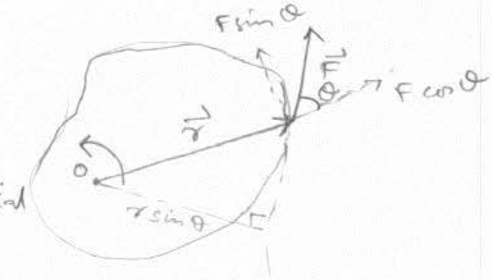
$$K = \frac{1}{2}\, r\, 2\pi\, \rho\, (\omega r)^2$$

$$K = \frac{1}{\cancel{2}}\, \cancel{r}\, \cancel{2\pi}\, \frac{M(\omega r)^2}{\cancel{2\pi}\cancel{r}} = \frac{1}{2}(Mr^2)\,\omega^2 = \frac{1}{2} I \omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

Torque

$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$\tau = r F \sin\theta$$

$\tau = r(F\sin\theta) \rightarrow$ r times tangential force

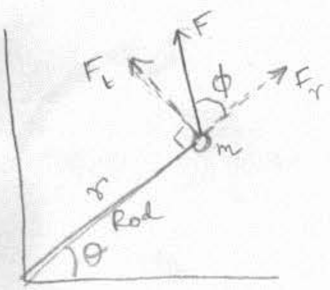
or $\tau = (r\sin\theta)F \rightarrow$ force times ⊥ dist from axis of rotation

Newton's Second Law of rotation

$$\boxed{\tau_{net} = I\alpha}$$

How to get this.

$\tau = F_t r$ $\qquad F_t = m a_t$



$$\tau = m a_t r = m(\alpha r) r = (mr^2)\alpha = I\alpha$$

$$\tau_{net} = I\alpha$$

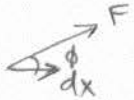
Just like Newton's second law for force.

Work and Rotational K.E.

$$\Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$$

$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = W$ (Work K.E. theorem)

Now,

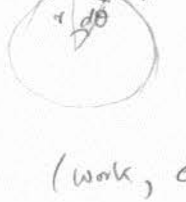


$dW = F dx \cos\phi$

$W = \int_{x_i}^{x_f} F\, dx \cos\phi$ (one dim. motion)

$$\therefore W = \int_{\theta_i}^{\theta_f} F(r\, d\theta)\cos\phi = \int_{\theta_i}^{\theta_f} (Fr\cos\phi)\, d\theta$$

$dx = r\, d\theta$



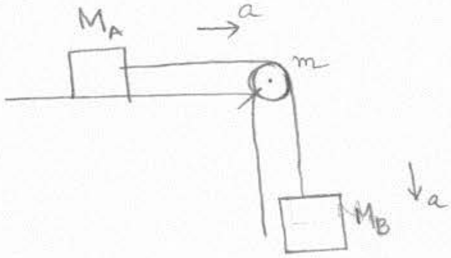
$$\boxed{W = \int_{\theta_i}^{\theta_f} \tau\, d\theta}$$

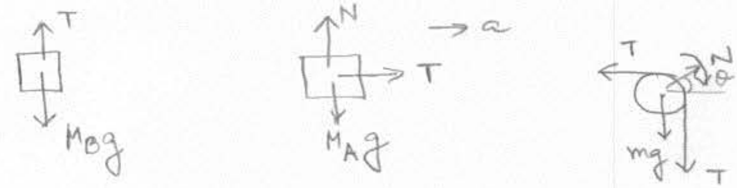
$W = \tau(\theta_f - \theta_i)$ (work, constant torque)

Power, $\dfrac{dW}{dt} = \tau\dfrac{d\theta}{dt} = \tau\omega$

Problem

Pulley has some mass m and radius R.





$M_B g = T$ $\qquad$ $T = M_A a$ $\qquad$ $T = N\cos\theta$

$mg + T = N\sin\theta$

Torque equation. $\left[\ TR - TR = 0.\right.$

When Torque is balanced

Problem: Unbalanced Torque

N & Mg pass through the center $\therefore$ No Torque

$$\boxed{TR = I\alpha}$$

Torque equation

$$mg - T = ma$$

find α

$$a = \alpha R$$

$$\alpha = \frac{TR}{I} = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR}$$

$$T = m(g-a)$$

$$\therefore \alpha = \frac{2m(g-a)}{MR} = \frac{2mg}{MR} - \frac{2m\alpha \cancel{R}}{M\cancel{R}} \Rightarrow \alpha\left[1 + \frac{2m}{M}\right] = \frac{2mg}{MR}$$

$$\therefore \alpha = \frac{2mg}{\cancel{M}R}\left[\frac{\cancel{M}}{M+2m}\right] = \frac{2mg}{(M+2m)R}$$

Let the disk start from rest at time $t=0$.

What is the rotational kinetic energy K at $t = t$ sec.

$$K_f = \frac{1}{2} I \omega^2$$

Need $\omega(t)$?

Initially at $t = 0$, $\omega(0) = 0$

$$\omega = \omega(0) + \alpha t \quad \Rightarrow \quad \omega = \alpha t$$

$$K = \frac{1}{2}\left(\frac{1}{2} M R^2\right)(\alpha t)^2 = \frac{1}{4} M (R\alpha t)^2$$

Corresponding relations for Translational and Rotational Motion

Pure Translation (Fixed direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = \frac{dx}{dt}$	Angular velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	Angular acceleration	$\alpha = \frac{d\omega}{dt}$
Mass	m	Rotational inertia	I
Newton's Second Law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\, d\theta$
Kinetic Energy	$K = \frac{1}{2} m v^2$	Kinetic energy	$K = \frac{1}{2} I \omega^2$
Power (const force)	$P = Fv$	Power (const torque)	$P = \tau\omega$
Work - Kinetic energy theorem	$W = \Delta K$	Work KE th	$W = \Delta K$

Calculating I (Moment of inertia)

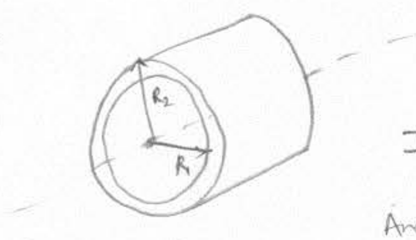
$$I = \sum m_i r_i^2$$

or $$I = \int_{\text{over mass}} dm\, r^2$$ (continuous body)

Some rotational Inertias

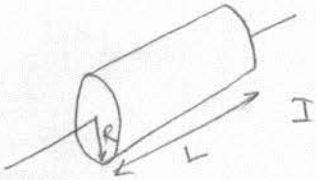
$I = MR^2$

Hoop about central axis

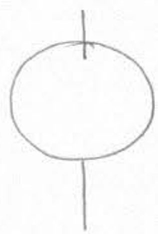


$I = \frac{1}{2}M(R_1^2 + R_2^2)$

Annular cylinder / ring about central axis

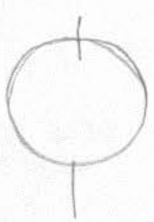


$I = \frac{1}{2}MR^2$



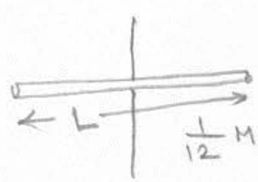
Solid sphere

$I = \frac{2}{5}MR^2$

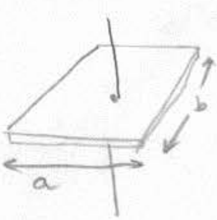


thin spherical shell

$I = \frac{2}{3}MR^2$



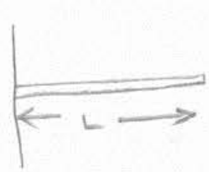
$\frac{1}{12}ML^2$

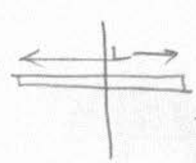


$I = \frac{1}{12}M(a^2 + b^2)$

Parallel axis theorem

$$I = I_{cM} + Mh^2$$

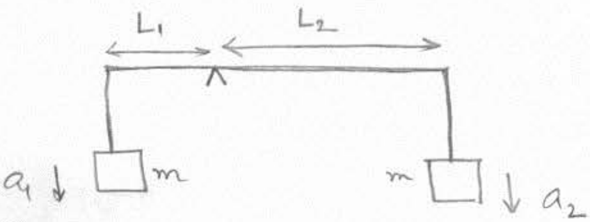




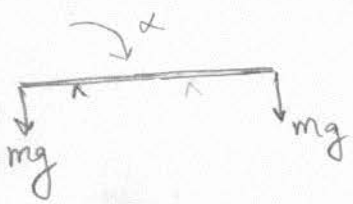
$I_{cm} = \frac{1}{12}ML^2$

$$I = I_{cm} + M\left(\frac{L}{2}\right)^2$$
$$= \left(\frac{1}{12} + \frac{1}{4}\right)ML^2 = \frac{1}{3}ML^2$$

Problem: Two blocks of equal mass m, suspended. The rod is held horizontally and then the fulcrum is released. What is the magnitude of initial acceleration of (a) closer block (b) other block.



What do you expect to happen?



$$mgL_2 - mgL_1 = I\alpha \quad \text{(Torqe eqn)}$$

$$a_2 = \alpha L_2 \qquad a_1 = \alpha L_1$$

$$I = mL_1^2 + mL_2^2 = m(L_1^2 + L_2^2)$$

$$\therefore \quad a_1 = \alpha L_1 = \frac{\cancel{m}g(L_2 - L_1)}{\cancel{m}(L_1^2 + L_2^2)} L_1 = \frac{g(L_2 - L_1)L_1}{L_1^2 + L_2^2}$$

and $$a_2 = \alpha L_2 = \frac{g(L_2 - L_1)}{(L_1^2 + L_2^2)} L_2$$