Simple harmonic motion (recap)

1) $x(t) = x_m \cos(\omega t + \underbrace{\phi}_{\text{initial phase}})$

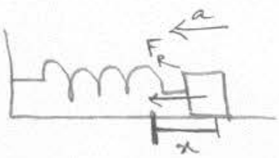
2) $v(t) = \dfrac{dx(t)}{dt}$

3) $a(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

4) $\omega = \sqrt{\dfrac{k}{m}}$

5) $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}}$

How do we start off with this?



$F_R = -kx$

$ma = -kx \Rightarrow m\dfrac{d^2x}{dt^2} = -kx$

(Newton's Law)

$\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x = -\omega^2 x \quad \therefore \quad \boxed{\dfrac{d^2x}{dt^2} + \omega^2 x = 0}$

Solution.

General solution: $x(t) = \underbrace{A\cos\omega t}_{} + \underbrace{B\sin\omega t}_{}$

linearly indep solutions superimposed.

$x(t) = A\cos\omega t + B\sin\omega t$

can be written as: if $A = x_m\cos\phi \quad B = -x_m\sin\phi$

$x(t) = x_m\cos(\omega t + \phi)$

Two arbitrary constants

x_m and ϕ are determined by the initial conditions which need to be mentioned to get the complete solution for $x(t)$

eg: at $t = 0$; $x(0) = x_0$ and $v(0) = 0$.

$\therefore x(0) = x_m\cos(\phi) = x_0$

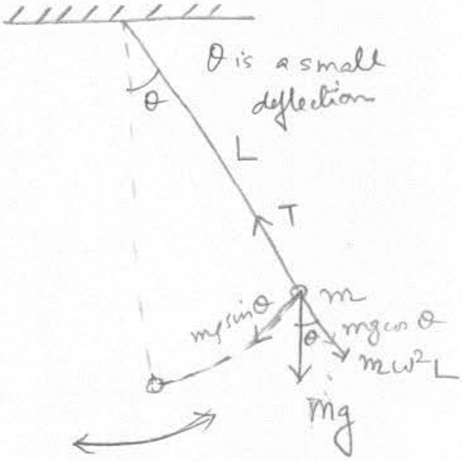
$v(t) = -x_m\omega\sin(\omega t + \phi)$; $v(0) = 0 = -x_m\omega\sin\phi$

$\therefore \phi = 0$ & $x_m = x_0$

$$\boxed{x(t) = x_0\cos\omega t}$$

Pendulums.

Simple pendulum.



$$T = m\omega^2 L + mg\cos\theta$$

$$\tau = -(mg\sin\theta)L \qquad = -I\,\alpha(t)$$

for small θ, $\sin\theta \approx \theta$

$$\tau = -(mgL)\,\theta \Rightarrow \overset{\text{compare}}{\tau(t) = -K\,\theta(t)}$$

$$K = mgL$$

Again

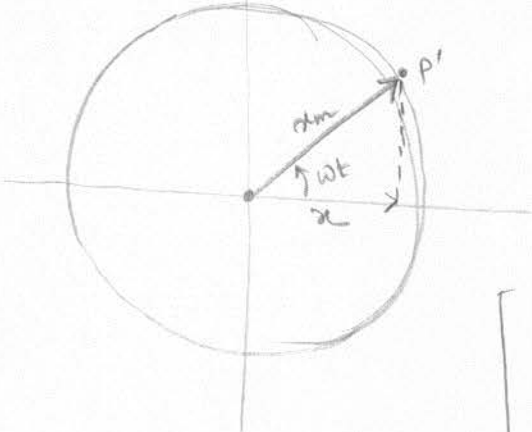
$$\underbrace{K = I\,\omega^2}_{\text{As in last lecture}} \Rightarrow \omega^2 = \frac{K}{I} = \frac{\cancel{m}g\cancel{L}}{\cancel{m}L^{\cancel{2}}} = \frac{g}{L}$$

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

Simple Harmonic and Uniform Circular motion.

Simple harmonic motion is the projection of uniform circular motion on a diameter of a circle in which the latter motion occurs.



$$\vec{r} = x_m \cos\omega t\ \hat{i} + x_m \sin\omega t\ \hat{j}$$

Projection on xaxis : $x = x_m \cos\omega t$

$$\vec{v} = \frac{d\vec{r}}{dt} = -x_m\omega \sin\omega t\ \hat{i} + x_m\omega\cos\omega t\ \hat{j}$$

$v_x = -x_m\omega\sin\omega t$ (proj. on axis)

$$\vec{a} = \frac{d\vec{v}}{dt} = -x_m\omega^2\cos\omega t\ \hat{i} - x_m\omega^2\sin\omega t\ \hat{j}$$

$a_x = -x_m\omega^2\cos\omega t$ (proj. along x-axis)

Differential equation for SHM.

Newtons law: $F = ma$

$F \downarrow$ Restoring force of spring

$$-kx = m\frac{d^2x}{dt^2} \Rightarrow \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0.$$

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

$\frac{k}{m} = \omega^2$ as earlier.

$\downarrow$ solve it we get

$\nearrow$ General solution

$$x(t) = \underbrace{x_m}_{1\ \text{constant}} \cos(\omega t + \underbrace{\phi}_{\text{second const.}})$$

Damped SHM.

extra damping force proportional to velocity and opposing it.

$$F = ma \Rightarrow \underbrace{F_R}_{\text{restoring}} + \underbrace{F_d}_{\text{damping}} = m\frac{d^2x}{dt^2} \qquad F_R = -kx$$

$$F_d = -bv = -b\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

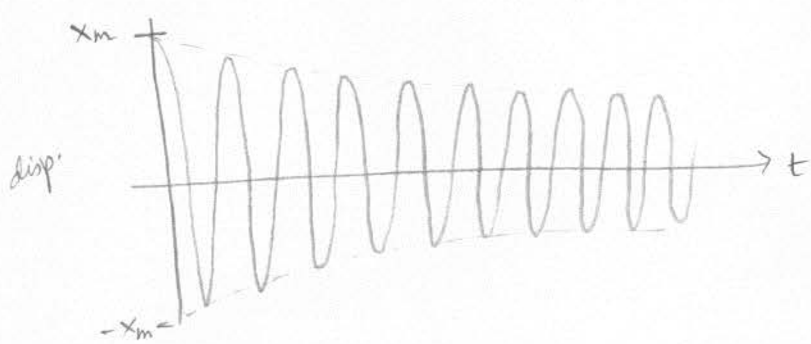
$$\frac{d^2x}{dt^2} + \left(\frac{b}{m}\right)\frac{dx}{dt} + \frac{k}{m}x = 0 \Rightarrow \frac{d^2x}{dt^2} + \left(\frac{b}{m}\right)\frac{dx}{dt} + \omega^2 x = 0.$$

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\omega - \frac{b^2}{4m^2}}$$

approximately

$$E(t) \approx \frac{1}{2}k x_m^2 \underbrace{e^{-bt/m}}_{\text{exponential decay of energy lost to damping force}}$$



Forced oscillation.

periodic force acting on system $F_p(t) = F_p \cos(\omega_p t + \phi)$

$$m \frac{d^2x}{dt^2} + Kx = F_p \cos(\omega_p t + \phi)$$

$$\frac{d^2x}{dt^2} + \omega^2 x = \left(\frac{F_p}{m}\right) \cos(\omega_p t + \phi)$$

i) Natural frequency ω
ii) frequency of periodic force ω_p

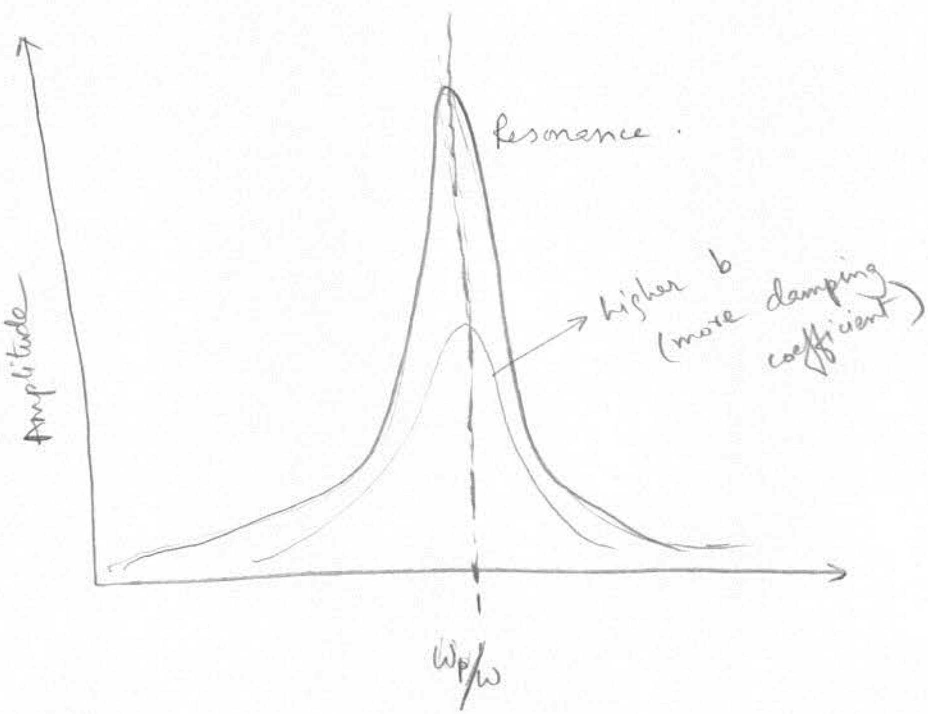
Resonance, $\omega_p = \omega$

Example: swing being pushed.

Compare: Mismatch of the push

Matched (angular) frequency of push

Highest amplitude attained at resonance



Waves.

Examples: i) when a stone is dropped on lake water
ii) when you shake a string

Types of Waves: (i) Mechanical waves
they involve a material medium and they propagate due to oscillations of the particles of the medium

(ii) Electromagnetic waves

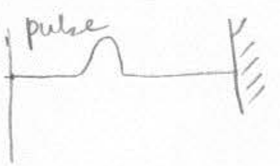
(iii) Matter wave: eg: e^-, p fundamental particles.

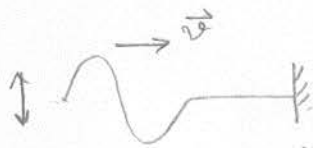
(i) Transverse wave
eg: ripples of water
string pulled

(ii) Longitudinal wave
eg: slinky /spring
Sound waves in air

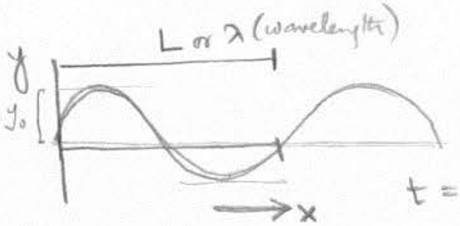
What is a wave?

Energy (or disturbance) propagates through a medium in the form of waves.





repeated periodic pulses form waves



Time snapshot

t = some particular, t_0 time

$y(x) = y_0 \sin kx$

$k = \frac{2\pi}{\lambda}$

single particle position snapshot.

Harmonic oscillator

$y(t) = y_0 \sin \omega t$ where $\omega = \frac{2\pi}{T}$

Superimpose both motions together

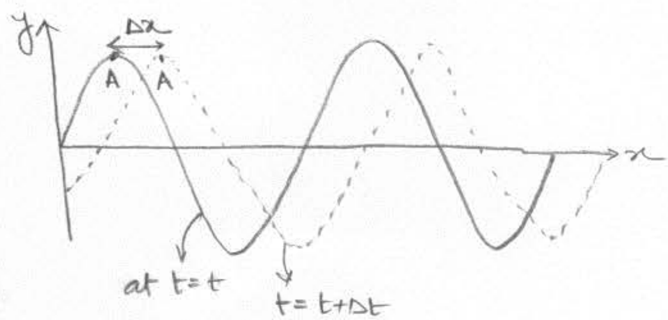
$$y(x,t) = y_m \sin(kx - \omega t)$$

y_m → amplitude; k → angular wavenumber; ω → angular frequency.

$$k = \frac{2\pi}{\lambda} \quad , \quad \omega = \frac{2\pi}{T}$$

λ → wavelength; T → timeperiod.

Superimposition of space & time pictures:



$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

The point A at $t=t$ has y displacement, $y = y_m \sin(kx - \omega t)$

" " A at $t = t+\Delta t$ " " " , $y = y_m \sin(k(x+\Delta x) - \omega(t+\Delta t))$

$$= y_m \sin\big(kx - \omega t + \underbrace{(k\Delta x - \omega \Delta t)}_{\text{has to be zero for small } \Delta x \,\&\, \Delta t}\big)$$

$$k\Delta x - \omega \Delta t = 0$$

$$\Rightarrow \quad k\Delta x = \omega \Delta t$$

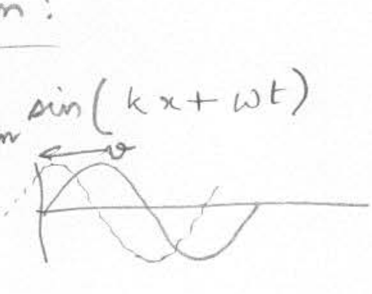
$$\boxed{v = \frac{\Delta x}{\Delta t} = \frac{\omega}{k}}$$ propagation velocity of wave

Now,

$$v = \frac{\omega}{k} = \left(\frac{2\pi}{T}\right)\left(\frac{\lambda}{2\pi}\right) = \boxed{\frac{\lambda}{T}}$$ use, $T = \frac{1}{f}$ $\therefore \boxed{v = \lambda f}$

For a wave moving in the opposite direction:

$$v = \frac{(-\Delta x)}{\Delta t} = -\frac{\omega}{k}$$ and $y(t) = y_m \sin(kx + \omega t)$



example: $y(t) = 5 \sin(2x + 5t)$

(i) what is the amplitude? $y_m = 5$

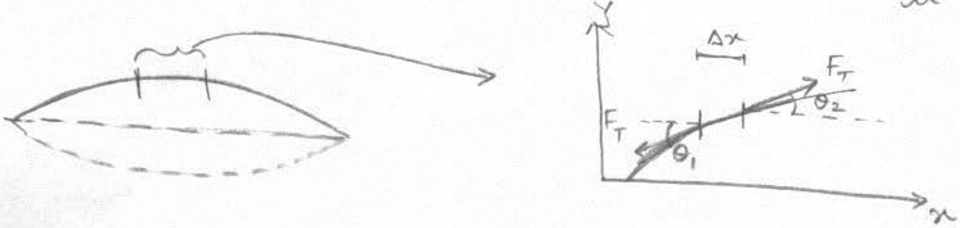
(ii) wavelength? $k = 2$ $\therefore \lambda = \frac{2\pi}{k} = \frac{\cancel{2}\pi}{\cancel{2}} = \pi$

(iii) time period? $T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$

$\omega = 5$

(iv) direction of propagation? $\leftarrow$

The Wave Equation.



μ is mass per unit length (linear density)

Force balance for element Δx,

$\Sigma F_x = 0$ & $\Sigma F_y = m a_y$

$$F_T \sin\theta_2 - F_T \sin\theta_1 = (\mu \Delta x) \frac{\partial^2 D}{\partial t^2}$$

D is displacement in y direction from mean position

θ_1 and θ_2 are small angles for small deflection

$$\sin\theta_1 \approx \tan\theta_1 = \frac{\partial D}{\partial x}\Big| = s_1$$

$$\sin\theta_2 \approx \tan\theta_2 = \left(\frac{\partial D}{\partial x}\right) = s_2$$

for small θ

$\sin\theta \approx \tan\theta = \frac{\partial D}{\partial x}$

$$\therefore \quad F_T (s_2 - s_1) = \mu \Delta x \frac{\partial^2 D}{\partial t^2} \Rightarrow \lim_{\Delta x \to 0} F_T \left(\frac{\Delta s}{\Delta x}\right) = \mu \frac{\partial^2 D}{\partial t^2}$$

$$F_T \frac{\partial}{\partial x}\left(\frac{\partial D}{\partial x}\right) = \mu \frac{\partial^2 D}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 D}{\partial x^2} = \left(\frac{\mu}{F_T}\right) \frac{\partial^2 D}{\partial t^2}$$

Now, dimention of $\left[\frac{\partial^2 D}{\partial x^2}\right] = \frac{L}{L^2} = L^{-1}$ $\therefore \left[\frac{\mu}{F_T}\right] = \frac{L^{-1}}{L T^{-2}} = L^{-2} T^{-2} = \left[\frac{1}{v^2}\right]$

$$\left[\frac{\partial^2 D}{\partial t^2}\right] = \frac{L}{T^{2}} = L T^{-2}$$

So,

$$\boxed{\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2}}$$ The wave equation in 1D

for wave propagating through a string : $v^2 = \frac{F_T}{\mu} \Rightarrow v = \sqrt{\frac{F_T}{\mu}}$

Solution for the wave equation :

if $v = \frac{\omega}{k} \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$

$$\frac{1}{k^2}\frac{\partial^2 y}{\partial x^2} = \frac{1}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

Check : $y(x,t) = y_m \sin(kx - \omega t)$

$\frac{\partial^2 y}{\partial x^2} = -y_m k^2 \sin(kx - \omega t)$ $\qquad \frac{\partial^2 y}{\partial t^2} = -y_m \omega^2 \sin(kx - \omega t)$.

$\therefore \frac{1}{k^2}\frac{\partial^2 y}{\partial x^2} = \frac{1}{\omega^2}\frac{\partial^2 y}{\partial t^2}$ is satisfied

Actually, $\sin(kx - \omega t)$ and $\cos(kx - \omega t)$ are both linearly independant solutions.

General solution: $y(x,t) = A \sin(kx - \omega t) + B \cos(kx - \omega t)$

Two arbitrary constants.

$y(x,t) = y_m \sin(kx - \omega t + \phi)$

$y_m \rightarrow$ amplitude.

$\phi \rightarrow$ initial phase.