Motion

A car is in motion with respect to a stationary house, which moves along with the rotating Earth. The Earth moves around the Sun. Studying the motion of different objects is called kinematics

As we saw above that motion can be really complicated, so let us start understanding the simplest examples of kinematics.

We consider now

(1) motion along a straight line only. The line may be vertical or slanted or horizontal, but it must be straight

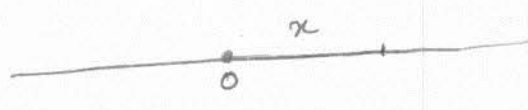
(2) in kinematics we don't think in terms of forces. All we consider is velocity or rate of change of velocity (acceleration)

(3) the moving particle is a point particle

[Note : Any rigid body motion can be considered as a superposition of motion of the center of mass and rotation about the center of mass]

POSITION AND DISPLACEMENT

It is always a good idea to choose a coordinate system. Once we choose an origin the position of a particle can be said to be x from the origin.



Now, the variable 'x' can be positive or negative, which means that the particle can be placed at any position compared to the origin.

Suppose, the position of the particle is changed from an initial position x_1 to a final position x_2 then the displacement of the particle is given by,

$$\Delta x = \underset{\text{final position}}{x_2} - \underset{\text{initial position}}{x_1}$$

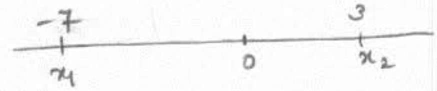
Thus, displacement of a particle is its change in position.

The distance moved by the particle is given by

$d = |x_2 - x_1|$. The distance is always a positive quantity.

Example: if $x_1 = -7$ m and $x_2 = 3$ m

Δx = displacement = $x_2 - x_1 = (3+7)$ m $= 10$ m



distance, $d = |x_2 - x_1| = |3+7| = 10$ m

So, we see that in this example the $\underset{\text{distance}}{\underset{\downarrow}{d}}$ and $\underset{\text{displ.}}{\underset{\downarrow}{\Delta x}}$ are same

example: if $x_1 = 2$ m, $x_2 = -7$ m.

then, $\Delta x = x_2 - x_1 = (-7-2)$ m $= -9$ m

$d = |x_2 - x_1| = |-7-2|$ m $= 9$ m

In 1 dimention the distance & displacement can differ by a sign.

The reason behind the difference is that the displacement tells us the net amount the particle has moved and its direction.

i.e. we move in the −ive x direction from 2 to −7 by 9 m.

Now suppose we move the particle from $x_2 = -7$ m towards the origin $(x_3 = 0)$ by 7 m. Then the net displacement is

$\Delta x = x_3 - x_1 = 0 - 2 = -2$ m

but the distance the particle moved is

$d = |x_3 - x_2| + |x_2 - x_1| = 7 + 9 = 16$ m

Definition: Average velocity, $v_{avg} = \dfrac{\text{total displacement}}{\text{total time}} = \dfrac{\Delta x}{\Delta t}$

Average speed, $S_{avg} = \dfrac{\text{total distance travelled}}{\text{total time taken}} = \dfrac{d}{\Delta t}$

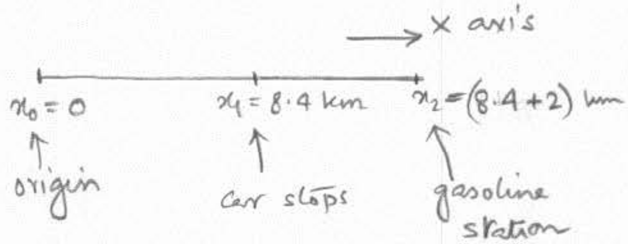
Average velocity and average speed.

ex: You drive a car along a straight road for 8.4 km at 70 km/h, where it runs out of gas and stops. Over the next 30 min, you walk another 2.0 km farther along the road to a gasoline station.

(a) What is your overall displacement from the beginning of your drive to the arrival at the station?

$\Delta x = x_2 - x_0 = (8.4+2) - 0 = 10.4$ km

(Total displacement = final position − initial pos.)



(b) What is the total time interval (Δt) from the beginning of your drive to the arrival at the station?

After the car stops you walk for 0.5 hour, $\therefore \Delta t_{walk} = 0.5$ hr

You drove the car for 8.4 km at avg velo, $v_{avg} = 70$ km/hr

$$\boxed{v_{avg} = \frac{\text{total displacement}}{\text{total time}} = \frac{\Delta x_{drive}}{\Delta t_{drive}}}$$

$$\Rightarrow \quad \Delta t_{drive} = \frac{\Delta x_{drive}}{v_{avg}} = \frac{8.4 \text{ km}}{70 \text{ km/hr}} = 0.12 \text{ hr}$$

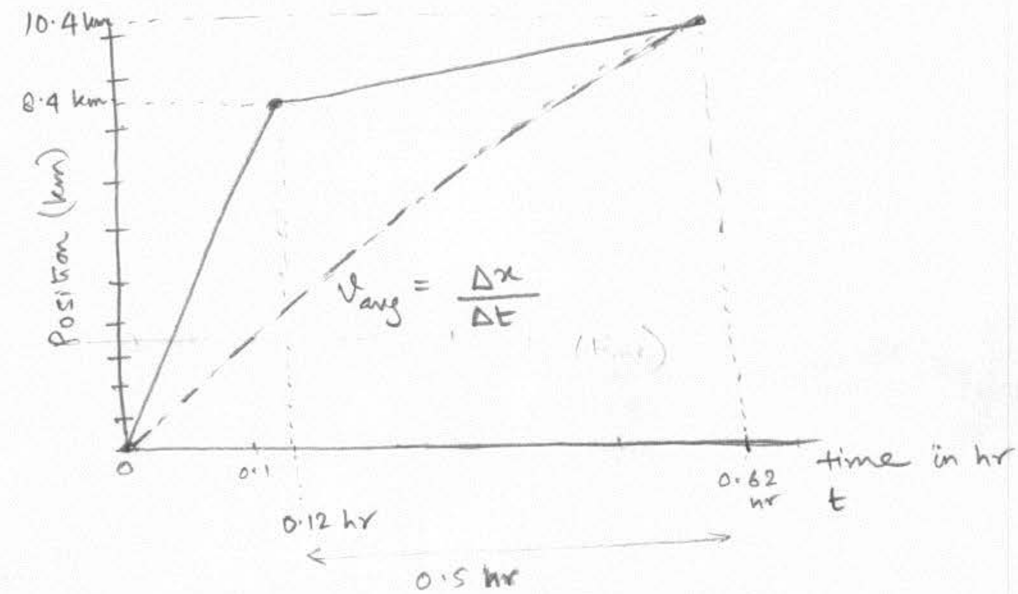
Total Time $\Delta t = \Delta t_{drive} + \Delta t_{walk} = (0.12 + 0.5)$ hr $= 0.62$ hr

(c) What is the average velocity from the beginning of your drive to arrival at station? Find both numerically and graphically.

$$v_{avg} = \frac{\text{total disp}}{\text{total Time}} = \frac{\Delta x}{\Delta t} = \frac{\Delta x_{dri} + \Delta x_{walk}}{\Delta t} = \frac{(8.4+2)\text{ km}}{0.62\text{ hr}} \approx 16.8\text{ km/hr}$$

$$= 17\text{ km/hr}$$

Graphically.



(d) Suppose at the pump you buy gas and walk back to the car in 45 min. What is the total average speed and the total average velocity?

$$S_{avg} = \frac{\text{total distance}}{\text{total time}} = \frac{(8.4 + 2 + 2)\text{ km}}{(0.12 + 0.5 + 0.75)\text{ hr}} = \frac{12.4\text{ km}}{1.37\text{ hr}}$$

$$= 9.1\text{ km/hr}$$

$$v_{avg} = \frac{\text{total displacement}}{\text{total time}} = \frac{8.4\text{ km}}{1.37\text{ hr}} = 6.1\text{ km/hr}$$

Problem solving.

(i) Understanding the problem

(ii) Write down given data and identify symbols
use symbols for calculation and plug numbers only at the end

(iii) Work problems with symbols, finally put numbers
see if units are ok!

you may need to make conversions.

(iv) is the answer reasonable, does it make sense?

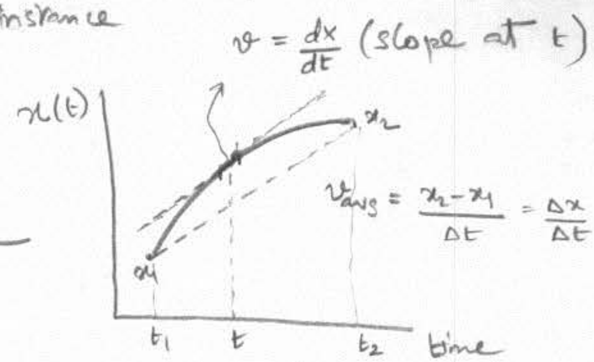
(v) Reading a graph of a function

Instantaneous Velocity and Speed

Instantaneous velocity v is velocity at an instant of time

we had, $v_{avg} = \dfrac{\Delta x}{\Delta t}$ ⟶ shrink Δt to zero to see an instance

$$v_{avg} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$



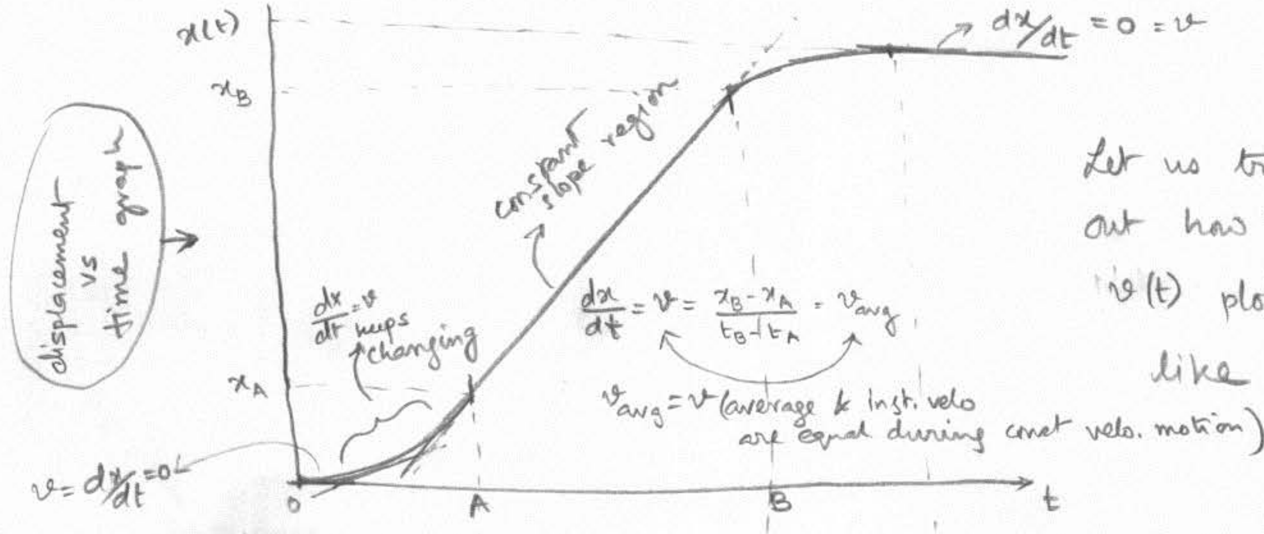
v is a vector quantity and has a direction
in 1D its just +ive or −ive

Speed is the magnitude of velocity.
just take off the direction from the velocity vector

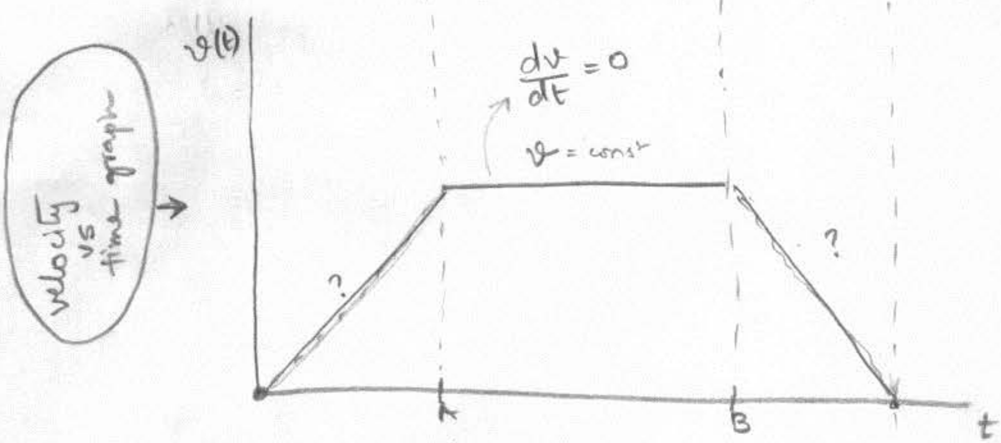
$v = +5$ km/hr or $v = -5$ km/hr both have speed $= 5$ km/hr
example, the speedometer of a car cannot know direction

Problem : example.

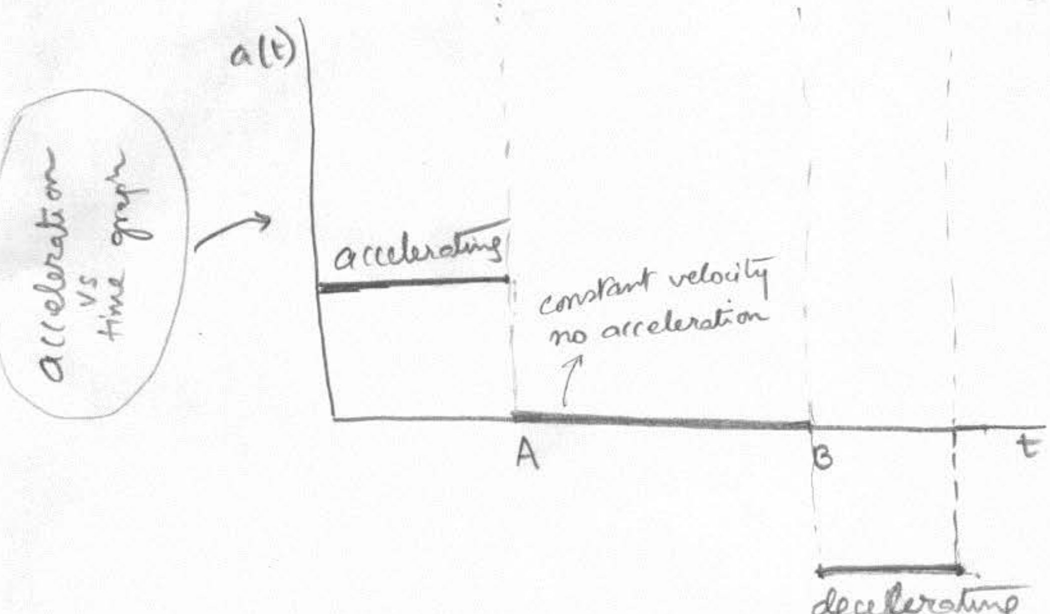
Consider the $x(t)$ plot for a car that starts from rest



Let us try to figure out how the corresponding $v(t)$ plot will look like



$v(t) = \frac{dx}{dt}$ (slope of the $x(t)$ plot)



What about $\frac{dv}{dt}$?

So, $\frac{dv}{dt} = a$ (acceleration)

rate of change of velocity

Same idea taken another level higher.

Can you predict the displacement time graph by looking at a velocity time graph ??

The idea of integration $\longleftrightarrow$ area under the curve.

if $v = \frac{dx}{dt}$ $\Rightarrow$ $v\,dt = dx$

integrating both sides.

$$\int_{t_i}^{t_f} v\,dt = \int_{x_i}^{x_f} dx$$

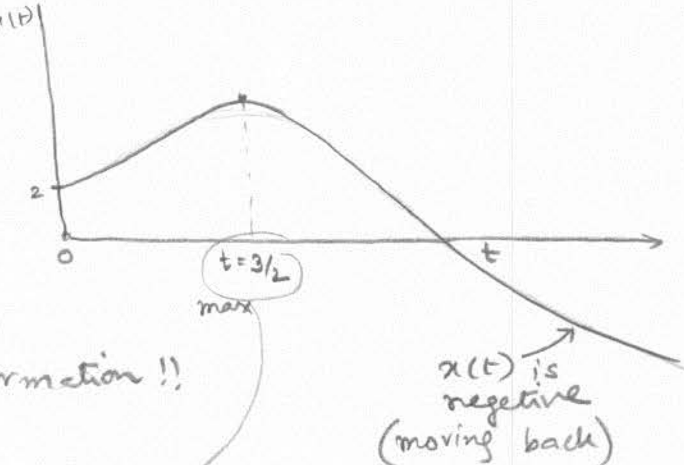
The velocity time graph tells you how v varies with time (i.e. $v(t)$)

using that,

$$\int_{t_i}^{t_f} v(t)\,dt = \int_{x_i}^{x_f} dx = \underbrace{(x_f - x_i)}_{\text{displacement}}$$

<u>example</u>:

$x(t) = 2 + 3t - t^2$ $\longrightarrow$



(i) find the instantaneous velocity.

(ii) find velocity at $t = 3/2$

(iii) find the instantaneous acceleration

(iv) from the instantaneous velocity, try getting $x(t)$. Do you need more information !!

(i) $v(t) = \frac{dx}{dt} = 3 - 2t$

(ii) at $t = 3/2$, $v(t) = 3 - 2 \cdot \frac{3}{2} = 0$ $\Rightarrow$ $\frac{dx}{dt} = 0$ [$x(t)$ at $t = 3/2$ is a maxima/minima]

$$\left.\frac{d^2x}{dt^2}\right|_{t=3/2} = -2 \text{ -ive} \therefore \text{maxima}$$

(iii) $a = \frac{dv}{dt} = -2$ ← constant retardation/deceleration

(iv) $v(t) = 3 - 2t$ $\Rightarrow$ $\int_{t_i}^{t_f} v(t)\,dt = x_f - x_i$ $\Rightarrow$ $x_f - x_i = 3(t_f - t_i) - 2\left.\frac{t^2}{2}\right|_{t_i}^{t_f}$

$$x_f - x_i = 3(t_f - t_i) - (t_f^2 - t_i^2)$$

$\boxed{\text{Boundary condition}}$ at $t_i = 0$, $x_i = 2$

$$x_f - 2 = 3(t_f - 0) - t_f^2 \Rightarrow \boxed{x = 2 + 3t - t^2}$$

The boundary condition is essentian to get the exact form of $x(t)$ from the function $v(t)$

ACCELERATION

When the velocity of a particle changes, it is undergoing acceleration

[Just like average velocity, $v_{avg} = \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{\Delta x}{\Delta t}$]

$$a_{avg} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

The (instantaneous) acceleration is $a = \lim\limits_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t} = \dfrac{dv}{dt}$

Rate of change of velocity.

- acceleration $a(t)$ is the slope of the velocity time graph (i.e. slope of $v(t)$)

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

Acceleration is second derivative of position, $x(t)$

units of acceleration: m/s^2 ; dimension: LT^{-2}

- If velocity is increasing with time $\leftrightarrow$ a is +ive

if velocity is decreasing with time $\leftrightarrow$ a is −ive

a is a vector quantity (both magnitude and direction)

[Observing Acceleration]

Consider yourself in a closed car (you can't see anything outside), can you tell if the car is at rest or in motion?

If the car is moving with a constant velocity, then you have no idea about the motion

If the car is accelerating / decelerating you feel a push backward/forward

Why does this happen ??

example: a particle's position

$x(t) = 4 - 27t + t^3$ x is in meters and t in sec.

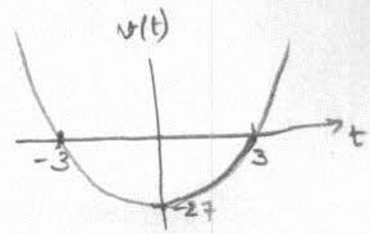
(i) find the particles velocity function & acceleration function

$v(t) = \frac{dx}{dt} = -27 + 3t^2$ (m/s) $\rightarrow$ Note: $[v] = \left[\frac{dx}{dt}\right] = \frac{L}{T} = LT^{-1}$

$a(t) = \frac{dv}{dt} = 6t$ (m/s²)

(ii) find the time when $v = 0$

$v(t) = 0 \Rightarrow -27 + 3t^2 = 0 \Rightarrow \underbrace{t = \pm 3\,s}_{2 \text{ solutions}}$



(iii) how does the particle move for $t \geq 0$?

Let us examine $x(t)$, $v(t)$ & $a(t)$ for $t \geq 0$

= At $t=0$, $x(0) = 4$ m , $v(0) = -27$ m/s , $a(0) = 0$

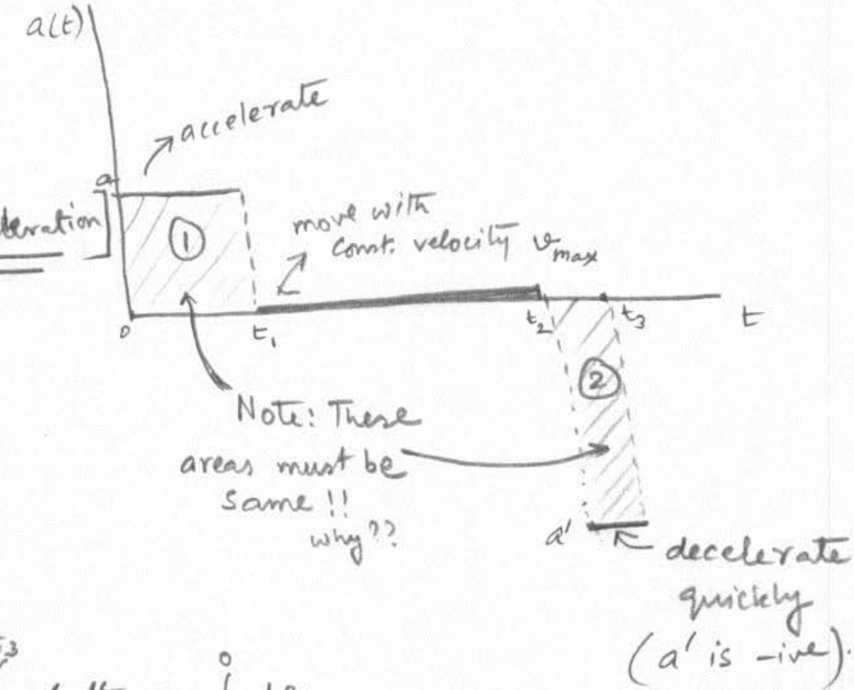
= for $0 < t < 3$ $v(t)$ is -ive but $a(t)$ is +ive

So it is accelerating but still moving in -ive X direction

the $v(3) = 0$ (velocity increases from -ive values and becomes zero at $t=3$; after which the velocity has +ive values so, particle changes direction of propagation at $t=3$)

Constant Acceleration : A special case

Motion of a car :



$$a = \frac{dv}{dt}$$

$$\int_0^{t_1} a\,dt = \int_0^{v_{max}} dv \quad \text{[During acceleration]}$$

$$a(t_1 - 0) = v_{max}$$

$$\boxed{a t_1} = v_{max}$$

→ area of block ①

[During retardation] : $\displaystyle\int_{t_2}^{t_3} a'\,dt = \int_{v_{max}}^{0} dv$

$$a'(t_3 - t_2) = -v_{max} \quad \Rightarrow \quad \boxed{v_{max} = -\underbrace{a'(t_3 - t_2)}}$$

→ area of block ②

∴ area of block ① = v_{max} = area of block ②

For constant acceleration :

Average acceleration = instantaneous acceleration

(just like we saw in the velocity case)

$$a = a_{avg} = \frac{v - v_0}{t - 0} \qquad v_0 = \text{initial velocity (at } t = 0)$$

$$\therefore \; at = v - v_0 \Rightarrow \boxed{v = v_0 + at} \; (\mathrm{I}) \quad (\text{generally: } v = u + at)$$

Indep of $(x - x_0)$

[NOTE: This formulae (I) and the others we derive are <u>only</u> applicable to <u>constant acceleration</u> cases.

Donot use them in problems where you don't know that the acceleration is constant] from (I) check : at $t=0$, $v = v_0$ (true)

$$\frac{dv}{dt} = a \quad \text{(true)}$$

Now, for the interval of time 0 to t

$$v_{avg} = \frac{x - x_0}{t - 0} \qquad x_0 \text{ is position at } t=0$$

$$\Rightarrow \quad v_{avg}\, t = x - x_0$$

Also, $v_{avg} = \frac{1}{2}(v_0 + v) = \frac{1}{2}(v_0 + \underbrace{v_0 + at}_{\text{from I}}) = v_0 + \frac{1}{2}at$

$$\therefore \quad \boxed{x - x_0 = v_0 t + \frac{1}{2}at^2} \ \text{(II)} \quad \left(\text{generally written as } s = ut + \frac{1}{2}at^2\right)$$

(indep. of v)

These equations involve $(x-x_0)$, v_0, v, a & t

All these equations describe the same phenomenon (constant acceleration motion)

We eliminate some parameters (that are not given in problem) and write different equations which are used to find the unknown parameter asked for in the problem.

Mutiply (II) by a on both sides

$$(x-x_0)a = at\left(v_0 + \frac{1}{2}at\right)$$

use $a = \frac{v - v_0}{t}$ (from I)

$$(x-x_0)a = \underbrace{\left(\frac{v-v_0}{\cancel{t}}\right)}_{\text{from I}}\cancel{t}\left(v_0 + \frac{1}{2}at\right)$$

Now: $v + v_0 = 2v_0 + at$ (from (I) add v_0 on both sides)

$$\frac{1}{2}(v+v_0) = v_0 + \frac{1}{2}at$$

$$= \frac{1}{2}(v - v_0)(v + v_0) = \frac{1}{2}(v^2 - v_0^2)$$

$$\therefore \quad \boxed{v^2 = v_0^2 + 2a(x - x_0)} \ \text{(III)} \ \text{(indep of } t\text{)}$$

example: Apply brake to a car from speed 100 km/h to 80 km/hr during a displacement of 88 m, at a constant acceleration (-ive here)

(a) What is a ?
(b) how much time is needed for the above decrease in speed ?

(a) You know, $v = 80$ km/hr , $v_0 = 100$ km/hr , $s = 88$ m

use $v^2 = u^2 + 2as \Rightarrow \dfrac{v^2 - u^2}{2s} = a$ [$\because u > v$, $v^2 - u^2$ is -ive $\therefore a$ is -ive]

$$a = -\frac{[u^2 - v^2]}{2s}$$ (plug values)

(b) find t :

$v = u + at \Rightarrow \dfrac{\overset{\text{-ive}}{(v-u)}}{\underset{\text{-ive}}{a}} = t$ [$\therefore$ t is +ive] <u>reasonable</u>