If a ball is projected vertically upward with a velocity v_0 what is the maximum hight it goes? and what time t it takes to reach that hight. ?

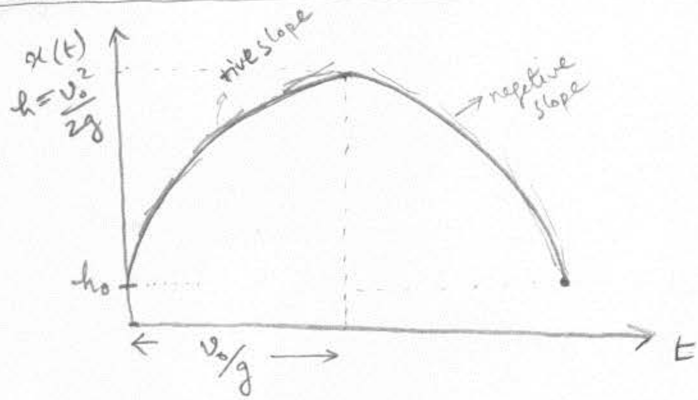
final velocity, $v = 0$.

use $v^2 = u^2 + 2as$ $\quad 0 = v_0^2 - 2gh \quad \Rightarrow \quad h = \dfrac{v_0^2}{2g}$

time taken to reach h,

use $v = u + 2at$ $\quad 0 = v_0 - gt \quad \Rightarrow \quad t = \dfrac{v_0}{g}$

Displacement time curve.



Getting $x(t)$ the functional form

$x(t)$ is hight at an instant t.

$$v^2 = v_0^2 - 2gx$$

$$x = \frac{v_0^2 - v^2}{2g} \quad \text{— (i)}$$

$$v = v_0 - gt$$

$$\Rightarrow v^2 = v_0^2 + g^2t^2 - 2v_0gt \text{ — (ii)}$$

$$x(t) = \frac{\cancel{v_0^2} - \cancel{v_0^2} - g^2t^2 + 2v_0gt}{2g} = -\frac{gt^2}{2} + v_0 t$$

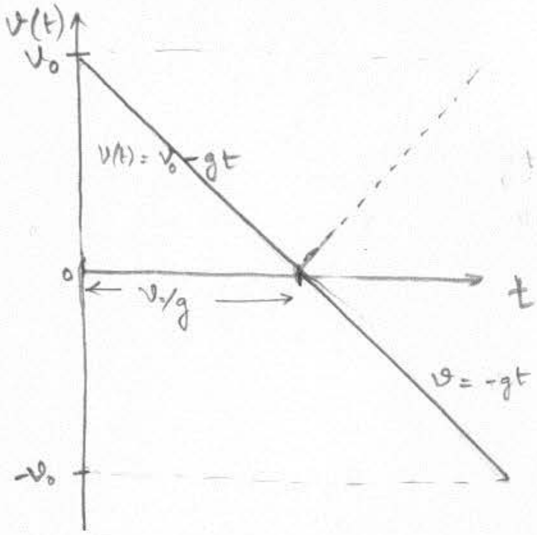
$$\therefore x(t) = -\frac{gt^2}{2} + v_0 t$$

During free fall:

$$v^2 = v_0^2 - 2g(x - x_0) \qquad (x_0 = h)$$

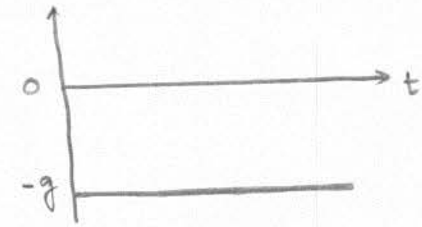
$$v = -gt \quad \Rightarrow \quad (x - x_0) = \frac{-g^{\cancel{2}}t^2}{2\cancel{g}} \quad \Rightarrow \quad x = h - \frac{gt^2}{2}$$

Velocity time graph



Upward velocity is +ive
downward velocity is -ive

Acceleration is constant



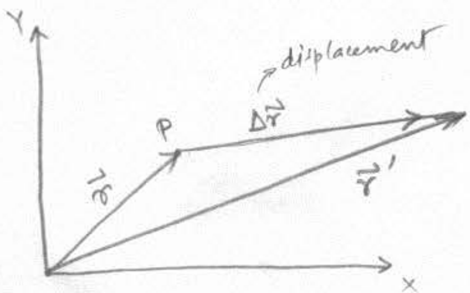
Kinematics in higher dimension

Vectors

Scalars :- physical quantities with magnitude & no direction sense
eg: distance, speed

Vectors :- physical quantities with magnitude & direction
[* quantities that follow the parallelogram law of vector addition]

example : position vetor & displacement.



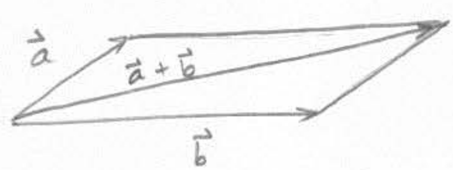
$\vec{r}$ is the initial position of P

$\vec{r}'$ is the final position

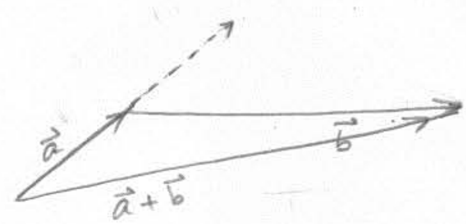
displacement, $\Delta\vec{r} = \vec{r}' - \vec{r}$

Adding vetors $\vec{a}$ and $\vec{b}$

Parallelogram law



Triangle law



Properties : (i) Commutivity $\quad \vec{a}+\vec{b} = \vec{b}+\vec{a}$

(ii) Associativity $\quad (\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(iii) Existance of inverse (-ive vector)

$\vec{a}+(-\vec{a}) = 0$ $\quad$ [Every vector has an inverse s.t. $\vec{a}+(-\vec{a})=0$]

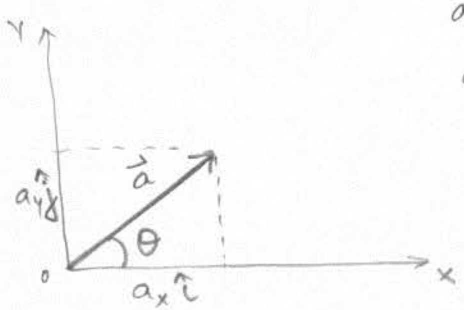
(iv) Scalar multiplication

$m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$

$|m\vec{a}| = m|\vec{a}|$ $\quad$ $(mn)\vec{a} = m(n\vec{a})$ $\quad$ $0\vec{a} = 0$

Components of a vector



a_x = projection of $\vec{a}$ on x axis

a_y = projection of $\vec{a}$ on y axis

$$\vec{a} = a_x \hat{i} + a_y \hat{j}$$

$$a_x = a\cos\theta \quad , \quad a_y = a\sin\theta$$

Magnitude / length of $\vec{a}$ is

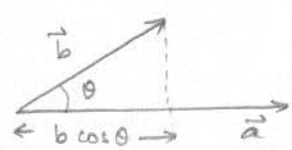
$$|\vec{a}| = \sqrt{a_x^2 + a_y^2}$$

angle with x-axis, θ

$$\tan\theta = \frac{a_y}{a_x}$$

Scalar product (dot product)

- $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

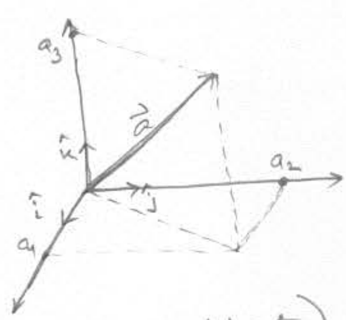


Projection of $\vec{b}$ on $\vec{a}$

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative)

- if $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$



$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Dot product of basis vectors (cartesian coordinate)

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \quad ; \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

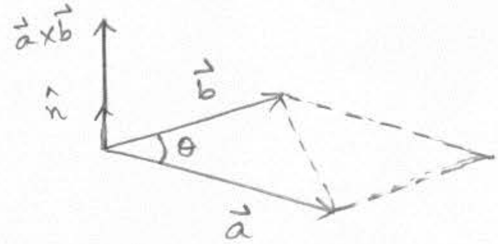
- $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = a \qquad |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = b$

$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{a\,b} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

- Unit vector in the direction of $\vec{a}$ is $\hat{a}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{a}$$

Vector product. (cross product)



$|\vec{a}\times\vec{b}|$ = Area of parallelogram

$|\vec{a}\times\vec{b}| = 2 \times$ Area of $\Delta = ab\sin\theta$.

- Right hand rule to find dir. of $(\vec{a}\times\vec{b})$

Direction of $(\vec{a}\times\vec{b})$ is $\perp$ to both $\vec{a}$ & $\vec{b}$ (i.e. it is $\perp$ to the plane containing $\vec{a}$ & $\vec{b}$)

* $\vec{a}\times\vec{b} = ab\sin\theta\,\hat{n}$

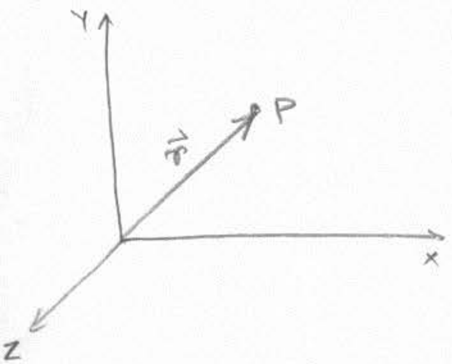
- $\vec{a}\times\vec{b}$ is anti-commutative i.e. $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

- $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$$

- $$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|} = \frac{|\vec{a}\times\vec{b}|}{ab}$$

Motion in higher dimension

• Position and displacement



P is at position $\vec{r}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Change in position is displacement

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k}$$

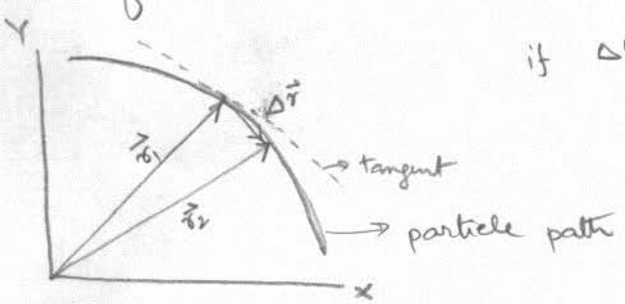
• Average velocity, $\vec{v}_{avg} = \dfrac{\Delta\vec{r}}{\Delta t} = \left(\dfrac{\Delta x}{\Delta t}\right)\hat{i} + \left(\dfrac{\Delta y}{\Delta t}\right)\hat{j} + \left(\dfrac{\Delta z}{\Delta t}\right)\hat{k}$

• Instantaneous velocity, $\lim\limits_{\Delta t \to 0} \dfrac{\Delta\vec{r}}{\Delta t} = \lim\limits_{\Delta t \to 0}\left(\dfrac{\Delta x}{\Delta t}\right)\hat{i} + \lim\limits_{\Delta t \to 0}\left(\dfrac{\Delta y}{\Delta t}\right)\hat{j} + \cdots$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} + \left(\frac{dz}{dt}\right)\hat{k}$$

$$\boxed{\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}}$$

Direction of instantaneous velocity ?



if $\Delta t \to 0$ then $(\Delta\vec{r})$ is in the direction of the tangent to the particle path.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$$

$\therefore$ $\vec{v}$ & $\Delta\vec{r}$ have the same direction

[The direction of $\vec{v}$ at every instant, is in the direction of the tangent to the particle's path at the particle's position.

- Average acceleration

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t} = \left(\frac{\Delta v_x}{\Delta t}\right)\hat{i} + \left(\frac{\Delta v_y}{\Delta t}\right)\hat{j} + \left(\frac{\Delta v_z}{\Delta t}\right)\hat{k}$$

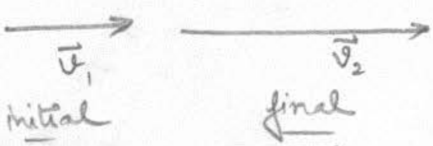
- Instantaneous acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{\Delta v_x}{\Delta t}\right)\hat{i} + \lim_{\Delta t \to 0}\left(\frac{\Delta v_y}{\Delta t}\right)\hat{j} + \dots$$

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}}$$

- The acceleration vector $\vec{a}$ is due to both, the change in magnitude and direction of $\vec{v}$.

- Acceleration due to change in magnitude of $\vec{v}$



(just like in 1D)

$$\boxed{\vec{v} = v(t)\,\hat{n}} \Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = \left(\frac{dv}{dt}\right)\hat{n}$$

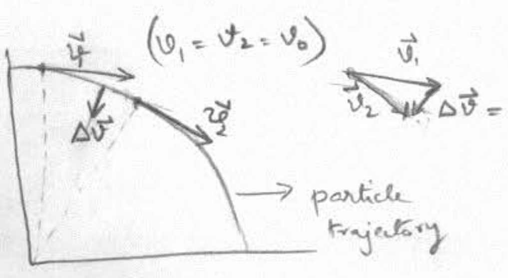
$\therefore$ $\underbrace{\vec{a}}_{\text{(component of } \vec{a} \text{ due to change in magnitude of } \vec{v}\text{)}}$ is in the same direction as $\vec{v}$

- Acceleration due to change in direction of $\vec{v}$

Consider $\vec{v} = v_0 \hat{n}$ $\quad$ v_0 is a constant (the magnitude of $\vec{v}$ is constant)
$\hat{n}$ is the direction of $\vec{v}$ which changes

$$\vec{a} = \frac{d\vec{v}}{dt} = v_0\left(\frac{d\hat{n}}{dt}\right)$$

$\vec{a}$ is in the radial direction of trajectory (i.e. direction of $\vec{a}$ is $\perp$ to $\vec{v}$ at every instant; just like direction of $\vec{v}$ is $\perp$ to $\vec{r}$ at every instant)



$\vec{a} = \frac{d\vec{v}}{dt}$ $\qquad$ $\vec{v} = v(t)\, \hat{n}(t)$

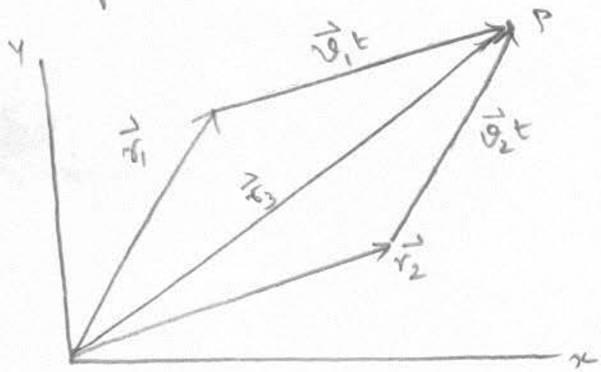
$$\boxed{\vec{a} = \frac{dv}{dt}\hat{n}(t) + v(t)\frac{d\hat{n}(t)}{dt}}$$

acceleration component in the direction of $\vec{v}$

acceleration component in the direction $\perp$ to $\vec{v}$

Problem example 1.

Two particles 1 and 2 move with constant velocities $\vec{v}_1$ and $\vec{v}_2$. At the initial moment their position vectors are $\vec{r}_1$ and $\vec{r}_2$. How must these four vectors be interrelated for the particles to collide?



For collision, after some time t they must be at same position call it P ($\vec{r}_3$)

$$\vec{r}_3 = \vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$$

Eliminate t and get the relation

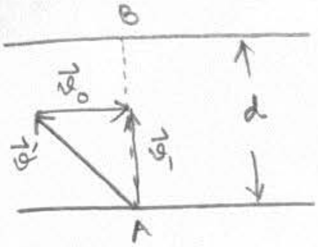
$$\vec{r}_1 - \vec{r}_2 = (\vec{v}_2 - \vec{v}_1) t \quad \Rightarrow \quad t = \frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|}$$

$$\therefore \boxed{\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|}}$$ is the required relation

Problem example 2

Two swimmers leave point A on one bank of the river to reach point B lying right across on the other bank. One of them crosses the river along the straight line AB while the other swims at right angles to the stream and then walks the distance that he has been carried away by the stream to get to point B. What is the velocity u of his walking if both swimmers reached the destination simultaneously? The stream velocity is $v_0 = 2.0$ km/hr and the velocity v' of each swimmer with respect to water equals 2.5 km/hr.

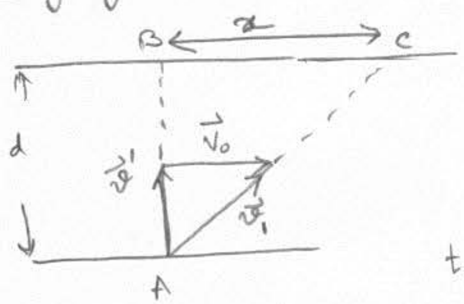
The swimmer (1) crosses along the straigh line AB



$$|\vec{v}_1| = \sqrt{v'^2 - v_0^2} \quad , \quad t = \frac{d}{\sqrt{v'^2 - v_0^2}}$$

(shortest path)

The swimmer (2) just swims $\perp$ to stream, so he is carried away by the stream



$$v_1 = \sqrt{v_0^2 + v'^2}$$

$$t_1 = \frac{d}{v'}$$

$$t_2 = \frac{x}{u} \longrightarrow \text{where } x = v_0 t_1 = v_0 \frac{d}{v'}$$

$$\therefore t_2 = \frac{v_0 d}{u v'} \quad \Rightarrow \therefore t = t_1 + t_2 = \frac{d}{v'} + \frac{v_0 d}{v' u}$$

both times equal $\therefore$

$$\frac{d}{\sqrt{v'^2 - v_0^2}} = \frac{d}{v'} + \frac{v_0 d}{v' u}$$

Solve for

$$u = \frac{v_0}{\left(1 - \frac{v_0^2}{v'^2}\right)^{-1/2} - 1}$$