PROJECTILE MOTION.

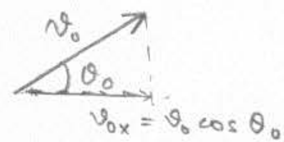
- The horizontal & vertical motions are independant of each other

Observed motion is resultant of both

- The Horizontal motion

$$(x - x_0) = v_{ox} t \qquad \text{(constant velocity motion)}$$

$$x - x_0 = (v_0 \cos\theta_0) t \longrightarrow (a)$$



- The Vertical motion

Same as a ball thrown upward with velocity v_{oy}

$$\rightarrow (y - y_0) = v_{oy} t - \frac{1}{2} g t^2$$

$$(y - y_0) = (v_0 \sin\theta_0) t - \frac{1}{2} g t^2 \quad - (1)$$

$$\rightarrow v_y = v_{oy} - g t$$

$$v_y = (v_0 \sin\theta_0) - g t \quad - (2)$$

$$\rightarrow v_y^2 = (v_0 \sin\theta_0)^2 - 2g(y - y_0) \quad - (3)$$

- Equation of path (trajectory)

Eliminate t from (1) using (a)

Use: $y_0 = 0$, $x_0 = 0$ (initial pts).

$$y = (\cancel{v_0} \sin\theta_0) \frac{x}{\cancel{v_0} \cos\theta_0} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

$$\boxed{y = (\tan\theta_0) x - \frac{g x^2}{2(v_0 \cos\theta_0)^2}}$$

Compare to $y = ax + bx^2$: Equation of a parabola

◦ Horizontal Range

$$R = (v_0 \cos\theta_0)\, t \quad -(1) \text{ horizontal}$$

$$0 = (v_0 \sin\theta_0)\, t - \tfrac{1}{2} g t^2 \quad -(2) \text{ vertical}$$

Eliminate t to get :

$$0 = (\cancel{v_0} \sin\theta_0) \frac{R}{\cancel{v_0} \cos\theta_0} - \frac{1}{2} g \left(\frac{R}{v_0 \cos\theta_0}\right)^2$$

$$0 = R \tan\theta_0 - \frac{1}{2} \frac{gR^2}{v_0^2 \cos^2\theta_0} \Rightarrow \tan\theta_0 - \frac{1}{2} \frac{gR}{v_0^2 \cos^2\theta_0} = 0 \quad (\because R \neq 0)$$

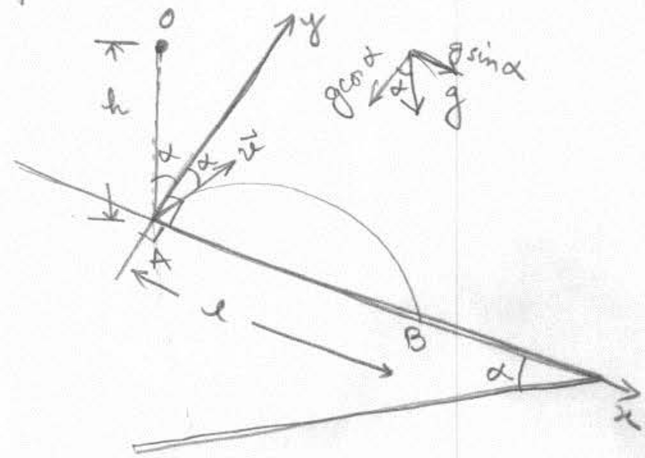
$$\Rightarrow \quad R = \frac{2 \tan\theta_0\, v_0^2 \cos^2\theta_0}{g} = (2 \sin\theta_0 \cos\theta_0) \frac{v_0^2}{g} = (\sin 2\theta_0) \frac{v_0^2}{g}$$

$$\boxed{R = \frac{v_0^2}{g} \sin 2\theta_0}$$

Horizontal range R is max for $\theta_0 = 45^\circ$

Problem

A ball starts falling with zero initial velocity on a smooth inclined plane forming an angle α with the horizontal. Having fallen the distance h, the ball rebounds elastically off the inclined plane. At what distance from the impact point will the ball rebound for the second time?



$v^2 = 2gh \Rightarrow v = \sqrt{2gh}$

for the fall OA

velocity in y direction $v\cos\alpha$

velocity in x direction $v\sin\alpha$

for motion along Y axis:

$$y = v\cos\alpha\, t - \frac{1}{2}(g\cos\alpha)t^2$$

Acceleration in y direction is $(-g\cos\alpha)$

Acc. in x dir. is $(g\sin\alpha)$

Suppose ball lands after time $\tau = 0$, (lands $\Rightarrow y=0$)

$$0 = v\cos\alpha\,\tau - \frac{1}{2}(g\cancel{\cos\alpha})\tau^2 \qquad (\tau \neq 0)$$

$$\boxed{\tau = \frac{2v}{g}}$$

distance travelled along x in τ time

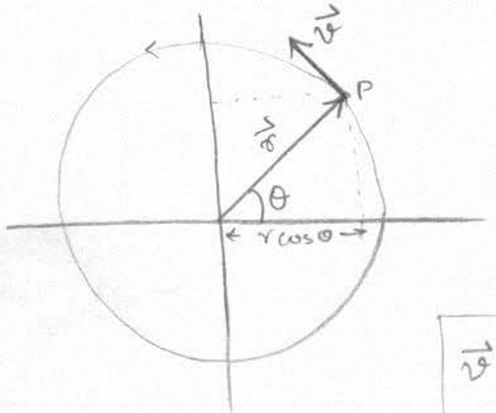
$$x = v\sin\alpha\,\tau + \frac{1}{2}(g\sin\alpha)\tau^2$$

$$\Rightarrow x = v\sin\alpha\frac{2v}{g} + \frac{1}{2}g\sin\alpha\left(\frac{2v}{g}\right)^2 = \frac{4\sin\alpha\, v^2}{g}$$

$\therefore$ the length, $l = \dfrac{4(2\cancel{g}h)\sin\alpha}{\cancel{g}} = 8h\sin\alpha$

$$\boxed{l = 8h\sin\alpha}$$ Answer.

Uniform circular motion



$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$\vec{v} = \frac{d\vec{r}}{dt} = -r\sin\theta\frac{d\theta}{dt}\hat{i} + r\cos\theta\frac{d\theta}{dt}\hat{j}$

Let angular velocity, $\omega = \frac{d\theta}{dt} \Rightarrow \int\omega\,dt = \theta$

$\omega t = \theta$

($\because \omega =$ const because uniform circular motion)

$$\boxed{\vec{v} = -r\omega\sin(\omega t)\hat{i} + r\omega\cos(\omega t)\hat{j}}$$

$\Rightarrow \boxed{v = \omega r}$ by taking Magnitude

Check : $\vec{v}\cdot\vec{r} = -r^2\omega\sin\omega t\cos\omega t + r^2\omega\sin\omega t\cos\omega t = 0$

$\therefore \vec{v} \perp \vec{r}$

Now, $\vec{a} = \frac{d\vec{v}}{dt} = -r\omega^2\cos(\omega t)\hat{i} - r\omega^2\sin(\omega t)\hat{j}$

$\Rightarrow \boxed{a = \omega^2 r = \frac{v^2}{r}}$ (use $v = \omega r$)

Check : $\vec{a}\cdot\vec{v} = 0$ [$\therefore \vec{a} \perp \vec{v}$ which is true because only direction of velocity is changing so no tangential acc. only radial acceleration]

Time period, $T = \frac{2\pi r}{v}$ [$\int\omega\,dt = \theta \Rightarrow \omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$

$\Rightarrow \boxed{T = \frac{2\pi r}{v}}$]

(3) A point moves in the xy plane according to the equation

$x = a\sin\omega t$, $y = a(1-\cos\omega t)$

where a and ω are positive const

(a) find the distance s traversed by the point during the time τ

(b) the angle between the point's velocity and acceleration vectors

Differentiating $x = a\sin\omega t$ $\quad y = a(1-\cos\omega t)$

$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \Rightarrow \quad v_x = a\omega\cos\omega t \quad v_y = a\omega\sin\omega t$

$\vec{v} = a\omega\cos\omega t\,\hat{i} + a\omega\sin\omega t\,\hat{j}$

$v = a\omega = \text{const.}$

acceler. $\vec{w} = \frac{d\vec{v}}{dt} = -a\omega^2\sin\omega t\,\hat{i} + a\omega^2\cos\omega t\,\hat{j}$

(a) The distance travelled in τ time

$$S = \int_0^{\tau} \underbrace{v}_{\text{mag. of velo}}\, dt = \int_0^{\tau} a\omega\, dt = a\omega\tau$$

(b) inner prod of $\vec{v}$ & $\vec{w}$

$\vec{v}\cdot\vec{w} = (a\omega\cos\omega t\,\hat{i} + a\omega\sin\omega t\,\hat{j})\cdot(a\omega^2\sin\omega t(-\hat{i}) + a\omega^2\cos\omega t(+\hat{j}))$

$= -a^2\omega^3\sin\omega t\cos\omega t + a^2\omega^3\sin\omega t\cos\omega t = 0$

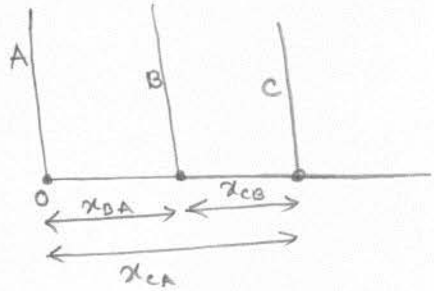
$\vec{v} \perp \vec{w}$ ($\frac{\pi}{2}$ is angle between them)

RELATIVE MOTION

(i) In One dimension

3 cars A, B and C.

car A is at rest, car B is at x_{BA} distance from A
car C is at x_{CB} distance from B
x_{CA} distance from A
} all at time t



$$\therefore x_{CA} = x_{CB} + x_{BA}$$

taking derivative

$$\frac{dx_{CA}}{dt} = \frac{dx_{CB}}{dt} + \frac{dx_{BA}}{dt}$$

$$v_{CA} = v_{CB} + v_{BA}$$

velocity of car C w.r.t A.

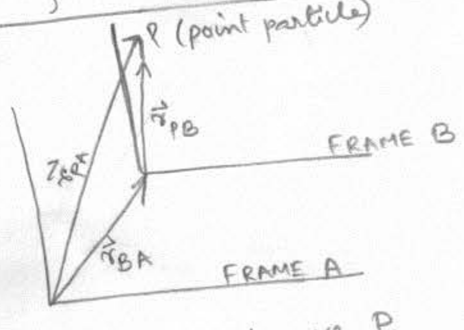
car C moving with velocity v_{CB} w.r.t. B

car B moving with velocity v_{BA} w.r.t A

Taking the derivative again

$$a_{CA} = a_{CB} \quad [\text{iff } v_{BA} \text{ is a constant}]$$

In higher dimension



Two frames, A & B. observe P.

$\vec{r}_{PB}$ is position of P wrt B
$\vec{r}_{PA}$ " " " P " A
$\vec{r}_{BA}$ " " " B " " → The frame B moves with const. velocity $\vec{v}_{BA}$ w.r.t. A.

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$

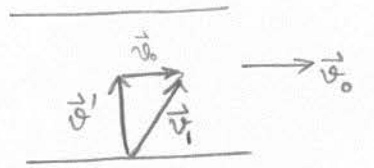
differentiate

$$\Rightarrow \boxed{\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}}$$

$$\vec{a}_{PA} = \vec{a}_{PB} \quad [\because \vec{v}_{BA} = \text{constant}]$$

Think about the swimmer problem we did in last class.

Swimmer (2)



If you are observing the swimmer 2 from a boat flowing with the stream velocity $\vec{v}_0$. What would you observe the swimmer doing?

To you the velocity of swimmer would be

$$\vec{v}_1 = \vec{v}_s + \vec{v}_0 \Rightarrow \vec{v}_s = \vec{v}_1 - \vec{v}_0 = \left(\vec{v}\,' + \vec{v}_0\right) - \vec{v}_0 = \vec{v}\,'$$

velocity w.r.t boat

velocity of boat w.r.t ground

velocity w.r.t ground.

NEWTON'S LAWS OF MOTION

from kinematics $\xrightarrow{\text{to}}$ Newtonian Mechanics

ask, what causes acceleration?

$\boxed{\text{FORCE}}$. breakthrough by Newton was because he could imagine a frictionless world.

Newtons first Law

Every body continues to remain in its state of rest or state of uniform motion in a straight line, unless a net external force is applied.

($\vec{F}_{net} = 0$, No resultant force)

Law of Inertia.

$\boxed{\text{Force}}$ causes acceleration; and if a body is accelerating that means there is an existing force

$\vec{F} \propto \vec{a}$ → the proportionality constant is inertia (or mass)

Inertial reference frame : frames at rest or moving with const. velocity.

Newton's Laws not valid in non inertial (i.e. accelerating) frames.

Newtons Second Law

$$\underbrace{\vec{F}}_{\text{Resultant force}} = m\vec{a} \longrightarrow F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = m(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$F_x = ma_1$$
$$F_y = ma_2$$
$$\vdots$$

The acceleration component along a given axis is caused only by the sum of the force components along the same axis, and not by force components along any other axis.

[The rate of change of momentum ($\vec{p} = m\vec{v}$) of a body is directly proportional to the external force applied, and the change takes place in the direction of the applied force] $\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + \vec{v}\underbrace{\left(\frac{dm}{dt}\right)}_{\text{generally } 0}$

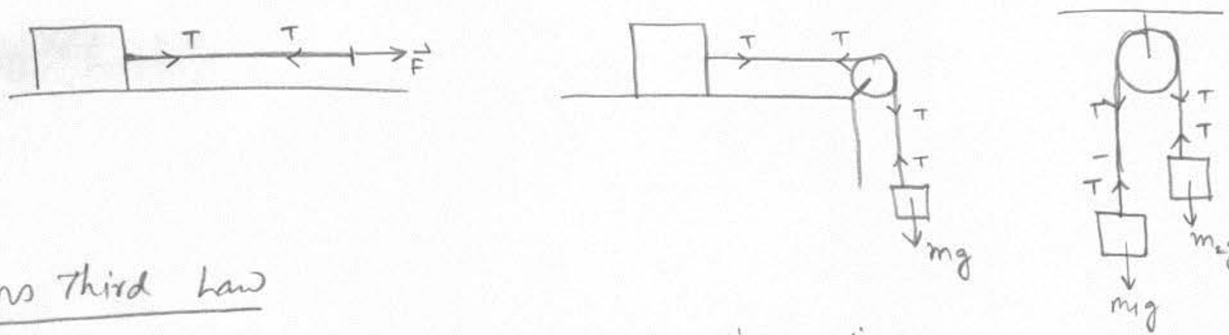
- Weight is the force with which the earth pulls a body

$$\vec{F} = m\vec{a} \quad \rightarrow \quad W = mg$$

- When a body presses against a surface, the surface pushes with a force $\vec{N}$ such that $\vec{N}$ balances all forces that push against the surface

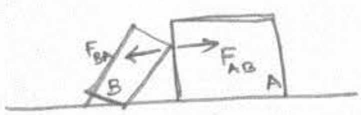
- Friction

- Tension



Newtons Third Law

Every action has an equal & opposite reaction force

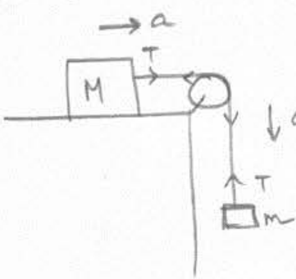


$$\vec{F}_{BA} = -\vec{F}_{AB}.$$

- Normal reaction force.

Solving problems

(1)



find the acceleration a?