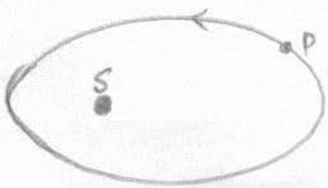
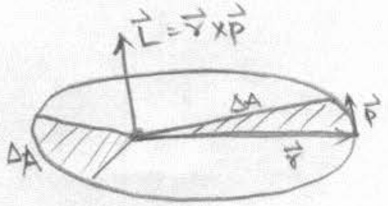
Kepler's Laws of Planetary motion

1st Law: The path of each planet about the Sun is an ellipse with the Sun at the focus



2nd Law: Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal periods of time.



Comes from conservation of $\vec{L}$ (ang mom.)

$$\vec{L} = \vec{r} \times \vec{p}$$

$$m\Delta A = \frac{1}{2}\, \vec{r} \times \vec{p}\, \Delta t = \frac{1}{2}\Delta t (\vec{r} \times \vec{p})$$

Motion on the same plane $(\vec{r} \times \vec{p})$ $\vec{L}$ is conserved (No external Torque)

3rd Law The ratio of the squares of the periods of any two planets revolving about the Sun is equal to the ratio of the cubes of their semimajor axis.

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{s_1}{s_2}\right)^3$$

s = semimajor axis

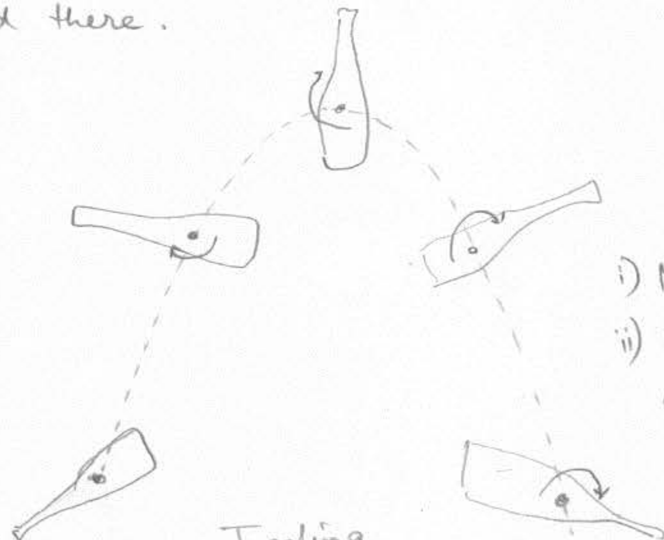
eg: consider circular motion

$$\frac{G m_1 M_s}{r_1^2} = \frac{m_1 v_1^2}{r_1^2} \qquad \text{and} \qquad v_1 = \frac{2\pi r_1}{T_1}$$

$$\frac{G m_1 M_s}{r_1^2} = m_1 \frac{4\pi^2 r_1}{T_1^2} \Rightarrow \frac{T_1^2}{r_1^3} = \text{const.} = \frac{\cancel{m_1} 4\pi^2}{G \cancel{m_1} M_s}$$

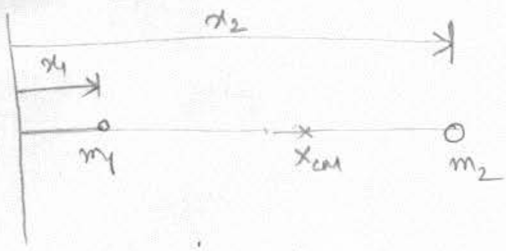
The Center of Mass

The center of mass of a body (or a system of bodies) is the point that moves as though all the mass were concentrated there and all external forces were applied there.



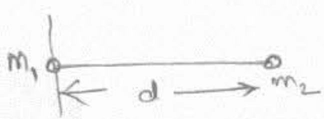
Juggling

i) Motion of Center of Mass
ii) Motion about center of mass (rotation)



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

[if m_1 is at $x_1 = 0$ then $x_{CM} = \dfrac{m_2 d}{m_1 + m_2}$]

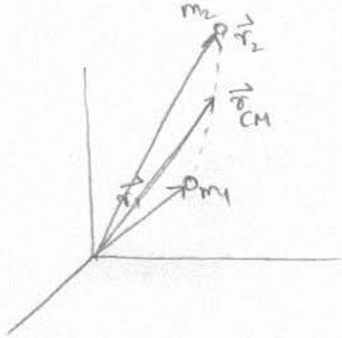


Generally for n bodies,

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

where, $\sum_{i=1}^{n} m_i = M$ (Total mass of system)

In higher dimension



$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

Now,
$$\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k}$$
$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$
$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

$$\therefore\quad x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k} = \frac{(m_1x_1+m_2x_2)\hat{i} + (m_1y_1+m_2y_2)\hat{j} + (m_1z_1+m_2z_2)\hat{k}}{(m_1+m_2)}$$

Thus, $x_{CM} = \dfrac{m_1x_1+m_2x_2}{m_1+m_2}$; $y_{CM} = \dfrac{m_1y_1+m_2y_2}{m_1+m_2}$; $z_{CM} = \dfrac{m_1z_1+m_2z_2}{m_1+m_2}$

Generally for n bodies

$$x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \qquad y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \qquad z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

(where, $M = \sum_{i=1}^{n} m_i$)

OR
$$\boxed{\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i}$$

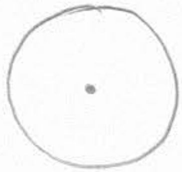
Continuous / solid Bodies.

$$x_{cm} = \frac{1}{M}\int x\,dm \qquad y_{cm} = \frac{1}{M}\int y\,dm \qquad z_{cm} = \frac{1}{M}\int z\,dm$$

density, $\rho = \dfrac{dm}{dV} = \dfrac{M}{V}$ (uniform density)

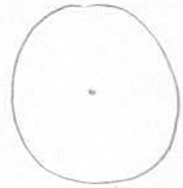
$$x_{cm} = \frac{1}{V}\int x\,dV \qquad y_{cm} = \frac{1}{V}\int y\,dV \qquad z_{cm} = \frac{1}{V}\int z\,dV$$

Center of Mass of standard shapes



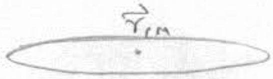
$\vec{r}_{cm}$ is at center

solid sphere



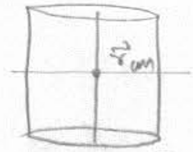
$\vec{r}_{CM}$ is at center

spherical shell

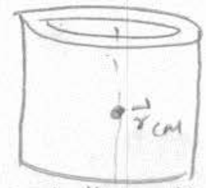


disk

uniform ring

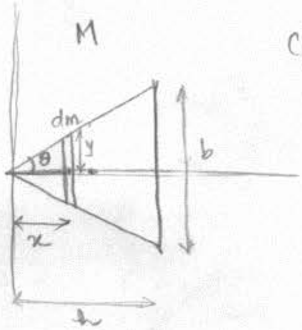


solid cylinder



hollow cylinder

Find CM of



$\theta = 30°$

$\tan\theta = \frac{y}{x}$

Choose coordinates such that $Y_{cM} = 0$, $z_{cm} = 0$

$$x_{cm} = \frac{\int x\, dm}{M}$$

$$dm = \rho(2y)\,dx = \rho(2x\tan\theta)\,dx = 2\rho\tan\theta\; x\,dx$$

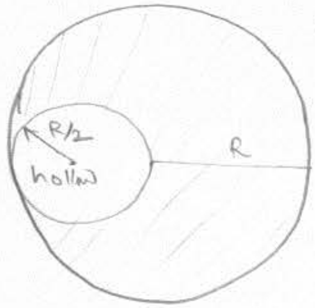
$$M = \left(\frac{1}{2}bh\right)\rho$$

$$x_{cm} = \frac{\int_0^h (2\rho\tan\theta)\, x^2\,dx}{\frac{1}{2}bh\rho} = \frac{2\rho\tan\theta}{\frac{1}{2}bh\rho}\; x^3/3\Big|_0^h$$

$$x_{cm} = \frac{4}{bh}\,\frac{1}{\sqrt{3}}\,\frac{h^3}{3} = \frac{4h^2}{b\,3\sqrt{3}}$$

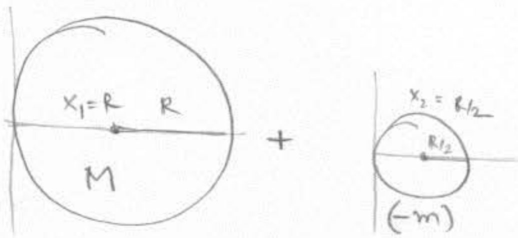
$$\vec{r}_{cM} = \frac{4h^2}{b\,3\sqrt{3}}\,\hat{i} + 0\,\hat{j}$$

Find CM of uniform disk of radius R



Method 1

$Y_{cm} = 0$



$$X_{cm} = \frac{x_1 M - x_2 m}{(M - m)}$$

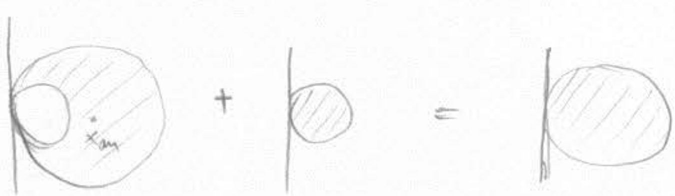
$$X_{cm} = \frac{RM - R/2\, m}{M - m} \qquad M = \pi R^2 \rho \qquad m = \pi \left(\frac{R}{2}\right)^2 \rho$$

$$X_{cm} = \frac{R\left[\cancel{\pi}\cancel{R^2}\cancel{\rho} - \frac{1}{2}\cancel{\pi}\frac{\cancel{R^2}}{4}\cancel{\rho}\right]}{\cancel{\pi}\left[1\cancel{R^2} - \frac{\cancel{R^2}}{4}\right]\cancel{\rho}} = \frac{R\left(1 - \frac{1}{8}\right)}{\left(1 - \frac{1}{4}\right)} = R\,\frac{7/8_2}{3/\cancel{4}} = \frac{7R}{6}$$

[Also

$$X_{cm}(M - m) + \frac{R}{2} m = RM.$$

$$X_M = \frac{RM - \frac{Rm}{2}}{M - m} = \frac{R\,\pi R^2 \rho - \frac{R}{2}\pi\left(\frac{R}{2}\right)^2 \rho}{\pi R^2 \rho - \pi (R/2)^2 \rho} = \frac{R\left(1 - \frac{1}{8}\right)}{1 - \frac{1}{4}} = \frac{7R}{6}$$]



Newton's Second Law for a System of particles

$$\vec{F}_{Net} = M\,\vec{a}_{CM} \qquad \text{[Translation of Center of Mass]}$$

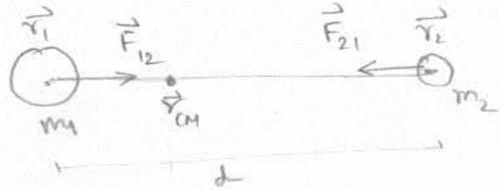
$\vec{F}_{Net}$ → Net Resultant force

M → Total mass

$\vec{a}_{CM}$ → acceleration of CM

If there is no net external forces, then there is no translation (/ change in position) of center of mass

eg:



Gravitational forces

$$F_{12} = \frac{G m_1 m_2}{d^2} = F_{21}$$

Initially, $\vec{r}_{CM} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$

finally, $\vec{r}_{CM} = \dfrac{m_1\vec{r}_1{}' + m_2\vec{r}_2{}'}{m_1+m_2}$

$\vec{r}_{CM\,initial} = \vec{r}_{CM\,final}$ [$\because$ No Net external force applied]

Linear momentum

$$\vec{p} = m\vec{v}$$

Newton's 2nd Law, rate of change of momentum is equal to the net force applied

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \left(\frac{dm}{dt}\right)\vec{v} + m\left(\frac{d\vec{v}}{dt}\right)$$

if $\dfrac{dm}{dt} = 0$ then $\vec{F}_{net} = \dfrac{d\vec{p}}{dt} = m\left(\dfrac{d\vec{v}}{dt}\right) = m\vec{a}$

Linear momentum of a system of particles

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i \quad \Rightarrow \quad M\vec{r}_{CM} = m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n$$

as if all the mass is at $\vec{r}_{CM}$

differentiat w.r.t t

$$M\left(\frac{d\vec{r}_{CM}}{dt}\right) = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 + \ldots + m_n\dot{\vec{r}}_n$$

[here $\frac{d\vec{r}_i}{dt} = \dot{\vec{r}}_i$ its a way to represent time derivatives]

$$M\dot{\vec{r}}_{CM} = \sum_{i=1}^{n} m_i\dot{\vec{r}}_i$$

$$\underbrace{\vec{P}_{CM}}_{\text{Momentum of CM}} = \underbrace{\sum_{i=1}^{n}\vec{P}_i}_{\text{sum of momenta of all masses.}}$$

$$\frac{d\vec{P}_{CM}}{dt} = M\ddot{\vec{r}}_{CM} = M\vec{a}_{CM} = \sum_{i=1}^{n}\frac{d\vec{P}_i}{dt} = \sum_{i=1}^{m} m_i\ddot{\vec{r}}_i = \sum_{i=1}^{n} m_i\vec{a}_i$$

$$\boxed{M\vec{a}_{CM} = \sum_{i=1}^{n} m_i\vec{a}_i}$$

Center of mass motion is combined motion of all m_i's with a_i as acceleration

Conservation of Linear momentum

if no external force applied to system

$\vec{P}$ = constant

$$\because \vec{F} = \frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} = \text{const} \Rightarrow \vec{P}_i = \vec{P}_f$$

$$\begin{pmatrix}\text{total linear momentum}\\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{total linear momentum}\\ \text{at some later time } t_f\end{pmatrix}$$