Kepler's Laws of Planetary Motion

1st Law: The path of each planet about the Sun is an ellipse with the Sun at the focus.

2nd Law: Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal periods of time.

\[ \vec{a} = \vec{r} \times \vec{p} \]

\[ m \Delta A = \frac{1}{2} \vec{r} \times \vec{p} \, \Delta t = \frac{1}{2} E_0 \, (\vec{r} \times \vec{p}) \]

Motion on the same plane \((\vec{r} \times \vec{p})\) is conserved (No external Torque).

3rd Law: The ratio of the squares of the periods of any two planets revolving about the Sun is equal to the ratio of the cubes of their semimajor axis.

\[ \left( \frac{T_1}{T_2} \right)^2 = \left( \frac{r_1}{r_2} \right)^3 \]

Example: Consider circular motion

\[ G \frac{m_1 M_s}{r_1^2} = m_1 \, v_1^2 \]

and \( v_1 = \frac{2 \pi r_1}{T_1} \)

\[ G \frac{m_1 M_s}{r_1^2} = m_1 \frac{4\pi^2 r_1}{T_1^2} \Rightarrow \frac{T_1^2}{r_1^3} = \text{const.} = \frac{4\pi^2}{G \, m_1 M_s} \]
The Center of Mass

The center of mass of a body (or a system of bodies) is the point that moves as though all the mass were concentrated there and all external forces were applied there.

1) Motion of Center of Mass
2) Motion about center of mass (rotation)

\[ X_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \]

If \( m_1 \) is at \( x_1 = 0 \) then

\[ X_{CM} = \frac{m_2 \cdot 0}{m_1 + m_2} \]

Generally, for \( n \) bodies,

\[ X_{CM} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]

where, \( \sum_{i=1}^{n} m_i = M \) (Total mass of system)
In higher dimension

\[ \vec{\tau}_{\text{CM}} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} \]

Now,

\[ \vec{x}_{\text{CM}} = x_{\text{CM}} \hat{i} + y_{\text{CM}} \hat{j} + z_{\text{CM}} \hat{k} \]
\[ \vec{x}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k} \]
\[ \vec{x}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \]

\[ x_{\text{CM}} \hat{i} + y_{\text{CM}} \hat{j} + z_{\text{CM}} \hat{k} = \frac{(m_1 x_1 + m_2 x_2) \hat{i} + (m_1 y_1 + m_2 y_2) \hat{j} + (m_1 z_1 + m_2 z_2) \hat{k}}{m_1 + m_2} \]

Thus,

\[ x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \]
\[ y_{\text{CM}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \]
\[ z_{\text{CM}} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} \]

Generally for \( n \) bodies

\[ x_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \]
\[ y_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i \]
\[ z_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i \]

(Where, \( M = \sum_{i=1}^{n} m_i \))

or

\[ \vec{\tau}_{\text{CM}} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{x}_i \]

Continuous / Solid Bodies.

\[ x_{\text{CM}} = \frac{1}{M} \int x \, dm \]
\[ y_{\text{CM}} = \frac{1}{M} \int y \, dm \]
\[ z_{\text{CM}} = \frac{1}{M} \int z \, dm \]

Density,

\[ \rho = \frac{dm}{dV} = \frac{M}{V} \quad \text{(uniform density)} \]

\[ x_{\text{CM}} = \frac{1}{V} \int x \, dV \]
\[ y_{\text{CM}} = \frac{1}{V} \int y \, dV \]
\[ z_{\text{CM}} = \frac{1}{V} \int z \, dV \]
Center of Mass of Standard Shapes

- **Solid sphere**: \( \bar{r}_{cm} \) is at center
- **Spherical shell**: \( \bar{r}_{cm} \) is at center
- **Disk**: \( \bar{r}_{cm} \) is at center
- **Uniform ring**: \( \bar{r}_{cm} \) is at center
- **Solid cylinder**: \( \bar{r}_{cm} \) is at center
- **Hollow cylinder**: \( \bar{r}_{cm} \) is at center

**Find \( CM \) of**

Choose coordinates such that \( y_{cm} = 0 \), \( z_{cm} = 0 \)

\[
x_{cm} = \frac{1}{M} \int x \, dm
\]

\[
dm = \rho (2y) \, dx = \rho (2 x \tan \theta) \, dx = 2 \rho \tan \theta \, x \, dx
\]

\[
M = \left( \frac{1}{2} bh \right) \rho
\]

\[
x_{cm} = \frac{1}{M} \int x (2 \rho \tan \theta) \, x \, dx = \frac{2 \rho \tan \theta}{\frac{1}{2} bh} \left[ \frac{x^3}{3} \right]_0^h
\]

\[
x_{cm} = \frac{4}{bh} \frac{1}{3} \frac{h^3}{3} = \frac{4 h^2}{3 b 3 \sqrt{3}}
\]

\[
\bar{r}_{cm} = \frac{4h^2}{b 3 \sqrt{3}} (1 + 0.5)
\]
Find CM of a uniform disk of radius $R$.

Method 1

$y_{CM} = 0$

$X_{CM} = \frac{x_1 M - x_2 m}{M - m}$

$X_{CM} = \frac{RM - R_2 m}{M - m}$

$M = \pi R^2 \rho$

$m = \pi \left( \frac{R}{2} \right)^2 \rho$

$X_{CM} = \frac{R \left[ \pi R^2 \rho - \frac{1}{2} \pi \left( \frac{R}{2} \right)^2 \rho \right]}{\pi \left[ 1 R^2 - \frac{R^2}{4} \right] \rho'}$

$X_{CM} = \frac{R \left( 1 - \frac{1}{8} \right)}{(1 - \frac{1}{4})} = \frac{7R^2}{3/4} = \frac{7R}{6}$

Also,

$x_{CM} (M-m) + \frac{R}{2} m = RM$

$x_M = \frac{RM - \frac{Rm}{2}}{M-m} = \frac{R \pi R^2 \rho - \frac{R}{2} \pi \left( \frac{R}{2} \right)^2 \rho'}{\pi R^2 \rho - \pi \left( \frac{R}{2} \right)^2 \rho'} = \frac{R \left( 1 - \frac{1}{2} \right)}{1 - \frac{1}{4}} = \frac{7R}{6}$
Newton's Second Law for a System of Particles

\[ F_{\text{net}} = \frac{M}{m} \cdot \ddot{x} \quad \text{[Translation of Center of Mass]} \]

Net Resultant force = Total mass \cdot acceleration of CM

If there is no net external force, then there is no translation (change in position) of center of mass

\[ \vec{F}_{12} \quad \vec{F}_{21} \quad \vec{F}_{1} \quad \vec{F}_{2} \]

Gravitational forces

\[ F_{12} = \frac{G m_1 m_2}{r^2} = F_{21} \]

Initially, \[ \vec{r}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \]

Finally, \[ \vec{r}'_{CM} = \frac{m_1 \vec{r}'_1 + m_2 \vec{r}'_2}{m_1 + m_2} \]

\[ \vec{r}_{CM} \; \text{Initial} = \vec{r}_{CM} \; \text{find} \quad \text{[\because No\; external\; force\; applied]} \]

Linear momentum

\[ \vec{p} = m \vec{v} \]

Newton's 2nd Law: Rate of change of momentum is equal to the net force applied

\[ F_{\text{net}} = \frac{dp}{dt} = \frac{d}{dt} (m \vec{v}) = m \frac{d\vec{v}}{dt} + \frac{dm}{dt} \vec{v} \]

If \[ \frac{dm}{dt} = 0 \quad \text{then} \quad F_{\text{net}} = \frac{dp}{dt} = m \frac{d\vec{v}}{dt} = m \ddot{x} \]
Linear momentum of a system of particles

\[ \dot{\vec{r}}_{CM} = \frac{1}{M} \sum_{i=1}^{n} m_i \dot{r}_i \quad \Rightarrow \quad \vec{M} \dot{\vec{r}}_{CM} = m_1 \ddot{r}_1 + m_2 \ddot{r}_2 + \cdots + m_n \ddot{r}_n \]

as if all the mass is at \( \vec{r}_{CM} \)

differentiate w.r.t. \( t \)

\[ M \left( \frac{d\vec{v}_{CM}}{dt} \right) = m_1 \ddot{r}_1 + m_2 \ddot{r}_2 + \cdots + m_n \ddot{r}_n \]

\[ M \ddot{\vec{r}}_{CM} = \sum_{i=1}^{n} m_i \ddot{r}_i \]

\[ \vec{P}_{CM} = \sum_{i=1}^{n} \vec{P}_i \]

\[ \text{Momentum of CM of all masses} \]

\[ \frac{d\vec{P}_{CM}}{dt} = M \ddot{\vec{r}}_{CM} = M \dot{\vec{a}}_{CM} = \sum_{i=1}^{n} \frac{d\vec{P}_i}{dt} = \sum_{i=1}^{n} m_i \ddot{r}_i = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i \]

\[ M \dot{\vec{a}}_{CM} = \sum_{i=1}^{n} m_i \dot{\vec{a}}_i \]

Center of mass motion is combined motion of all \( m_i \)'s with \( \dot{r}_i \) acceleration.

Conservation of linear momentum

if no external force applied to system

\[ \vec{P} \text{ constant} \]

\[ \vec{F} = \frac{d\vec{P}}{dt} = 0 \quad \Rightarrow \quad \vec{P} \text{ constant} \quad \Rightarrow \quad \vec{P}_i = \vec{P}_f \]

\[ \text{(initial linear momentum)} = \text{(final linear momentum)} \]

\[ \text{at some initial time } t_i \quad \text{at some later time } t_f \]