Approximation of Trigonometric Functions

Using the theory of Taylor series we can show that the following identities hold for all real numbers *x* (where *x* are angles in radians)

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots -\infty < x < \infty; \quad n \in \mathbb{N}$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots -\infty < x < \infty; \quad n \in \mathbb{N}$$

These identities are sometimes taken as the *definitions* of the sine and cosine function. They are often used as the starting point in a rigorous treatment of trigonometric functions and their applications, since the theory of infinite series can be developed from the foundations of the real number system, thus independent of any geometric considerations.



The sine function (blue) is closely approximated by its *Taylor polynomial* of degree 7 (pink) for a full cycle centered on the origin.

One of the most important applications of trigonometric series is for situations involving very small angles ($x \le 1$). For such angles, the trigonometric functions can be

approximated by the first term in their series. This gives the useful small angle approximations:

$$x << 1 \qquad \Rightarrow \qquad \begin{cases} \sin x \ \cong \ x \\ \cos x \ \equiv \ 1 \end{cases}$$

The approximation $\sin x = x$ reaches a 1% error at about 14°:

$$14^{\circ} = \frac{\pi}{180} 14^{\circ} \text{ rad} \qquad \sin\frac{\pi}{180} 14^{\circ} \cong 0.2443 \qquad \frac{\left|\sin\frac{\pi}{180} 14^{\circ} - 0.2443\right|}{\sin\frac{\pi}{180} 14^{\circ}} \cong 0.01$$

Examples of small angle approximation are in the calculation of

- the period of a simple pendulum,
- in most of the common expressions of geometrical optics that are built on the concept of **paraxial approximation** and **surface power** for lenses,
- the calculation of the intensity minima in **single slit diffraction**.