Chapter 30  Inductance, Electromagnetic Oscillations, AC circuits

Mutual Inductance

\[ \Phi_{21} = \text{flux created in 2 by field from 1} \]

- Flux \( \Phi \) \( \propto I \)
- \( N_2 \Phi_{21} \propto I_1 \)

\[ \frac{N_2 \Phi_{21}}{I_1} = \text{prop. constant, called Mutual Inductance} \]

\[ E_2 = -N_2 \frac{d\Phi_{21}}{dt} \quad \text{but} \quad \Phi_{21} = \frac{M_{21} I_1}{N_2} \]

\[ E_2 = -N_2 \frac{d}{dt} \left( \frac{M_{21} I_1}{N_2} \right) \]

\[ E_2 = -M_{21} \frac{dI_1}{dt} \]

\( M_{21} \) constant in time

\( E_2 \) depends on geometry

Likewise

\[ E_1 = -M_{12} \frac{dI_2}{dt} \]

it turns out \( M_{12} = M_{21} \)
\[ E_\text{1} = -M \frac{dI_2}{dt} \quad E_\text{2} = -M \frac{dI_1}{dt} \]

Unit for \( M \) is Henry \( \text{H} \)

\( 1 \text{H} = 1 \text{V} \cdot \text{s/A} = 1 \text{S} \cdot \text{s} \)

**Example:** Electric toothbrush charger

Given: base is solenoid of length \( L \), \( N_2 \) turns, and area \( A_2 \). The handle has \( N_1 \) turns.

Find the mutual inductance

\[ M = \frac{N_1 B A}{I} = \frac{N_1 B A}{I} \quad B = \mu_0 N_2 I \]

\[ M = \frac{\mu_0 N_2 N_1 A}{L} \]

**Self-Inductance**

As current increases, \( B \) created by current increases and so does flux \( (\Phi_B \) A\).

This induces an emf which resists the increase in current.

\[ \varepsilon = -N \frac{d\Phi_B}{dt} \]

\[ \varepsilon = -L \frac{dI}{dt} \]

Define \( L = \frac{N \Phi_B}{I} \Rightarrow \Phi_B = \frac{LI}{N} \)

A coil in a circuit is called an inductor
\[ I \quad \text{increasing} \quad \begin{array}{c}
\text{A} \\
\text{B}
\end{array} \]

\[ I \quad \text{decreasing} \quad \begin{array}{c}
\text{A} \\
\text{B}
\end{array} \]

which way is end?

\[ \text{a) like } - \quad \text{mains current flow in opposite direct} \]

\[ \text{b) like } - \quad \text{eg Tesla coil} \]

\[ \text{Energy Stored in a Magnetic Field} \]

\[ P = IU \quad \Rightarrow \quad P = IE = E \frac{dI}{dt} \]

\[ \text{d}W = P \, \text{d}t = E \, I \, \text{d}I \]

\[ \text{total energy} = \text{energy stored} \]

\[ W = \int \text{d}W = \int E \, I \, \text{d}I = \frac{1}{2} \int I^2 \quad \text{work done} = \text{energy stored} \]

\[ U = \frac{1}{2} LI^2 \quad \Rightarrow \quad \text{capacitive} \quad \text{V} \]

\[ U = \frac{1}{2} \left( \frac{N^2 \mu_0 A}{T} \right) \left( \frac{BL}{\mu_0 N} \right)^2 = \frac{1}{2} \frac{B^2}{\mu_0} \text{ (area) volume} \]

\[ U = \frac{1}{2} \frac{B^2}{\mu_0} \]
Induction in Circuits

Loop rule: $V_0 - L \frac{dI}{dt} - IR = 0$

"Charge"

recall $E = -L \frac{dI}{dt}$

$L \frac{dI}{dt} + IR = V_0$

separate variables:

$L \int_0^t \frac{dI}{V_0 - IR} = \int_0^t \frac{dt}{2}$

$\frac{1}{R} \ln(V_0 - IR) \bigg|_0^t = \frac{t}{2}$

$\ln\left(\frac{V_0 - IR}{V_0}\right) = -\frac{t}{2}$

$1 - \frac{IR}{V_0} = V_0 - \frac{IR}{V_0} = e^{-\frac{t}{\tau}}$

$I = \frac{V_0}{R} \left(1 - e^{-\frac{t}{\tau}}\right)$

when $\tau = \frac{L}{R}$
We can sketch: at $t=0$

\[ E = L \frac{dI}{dt} \]

\[-I = L \frac{dI}{dt} \]

\[-\int R \, dt = \int \frac{I}{L} \, dt \]

\[-\frac{R}{L} t = \ln \left( \frac{I}{I_0} \right) \]

\[ e^{-\frac{R}{L} t} = \frac{I}{I_0} \]

\[ I = I_0 e^{-\frac{R}{L} t} \]

Add capacitors \underline{LRC circuits}.

First consider idealized circuit - LC:

charged capacitor + inductor = \[
\text{switch closed @ } t=0
\]

work: \[-\frac{1}{L} \frac{dQ}{dt} + \frac{Q}{C} = 0 \]

\[ I = -\frac{dQ}{dt} \] charge leaving capacitor is creating the current.
\[-L \frac{d}{dt} \left( \frac{dQ}{dt} \right) + \frac{Q}{C} = 0 \quad \text{divide by} \ L\]
\[\frac{d^2 Q}{dt^2} + \frac{Q}{CL} = 0 \quad \text{2nd order diff eq Q} \]
\[\frac{d^2 Q}{dt^2} = -\frac{1}{CL} Q \]
\[\Rightarrow Q = Q_0 \cos(\omega t + \phi) \]

check: \[\frac{dQ}{dt} = Q_0 \omega (-\sin(\omega t + \phi)) \]
\[\frac{d^2 Q}{dt^2} = -Q_0 \omega^2 \cos(\omega t + \phi) \]
\[\frac{1}{LC} = \omega^2 \Rightarrow \omega = \sqrt{\frac{1}{LC}} \quad \text{frequency of the circuit} \]

\[
\begin{align*}
  \text{Charge on the capacitor oscillating} \\
  \begin{cases} 
    Q_0 & t > 0 \\
    -Q_0 & t < 0 
  \end{cases}
\end{align*}
\]

What about \( I \)?
\[I = -\frac{dQ}{dt} = -\frac{d}{dt} \left( Q_0 \cos(\omega t + \phi) \right) \]
\[= \omega Q_0 \sin(\omega t + \phi) \]
\[I = \omega Q \]

current in the inductor

current reversed - discharging, charging, etc. ... and on
What's happening to the energy?

\[ U_{cap} = \frac{1}{2} \frac{Q^2}{C} = \frac{Q_0^2}{2C} \cos^2(\omega t + \phi) \]

\[ U_{induct} = \frac{1}{2} LI^2 = \frac{Lw^2Q_0}{2} \sin^2(\omega t + \phi) = \frac{Q_0^2}{2C} \sin^2(\omega t + \phi) \]

Total energy

\[ U_{total} = \frac{Q_0^2}{2C} \cos^2(\omega t + \phi) + \frac{Q_0^2}{2C} \sin^2(\omega t + \phi) \]

\[ = \frac{Q_0^2}{2C} (\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)) \]

\[ = \frac{Q_0^2}{2C} \]

- constant

Energy Conservation

\[ \text{guess} \]

Real Life: LRC

Loop:

\[ -L \frac{dI}{dt} - IR + \frac{Q}{C} = 0 \]

\[ I = -\frac{dQ}{dt} \]

\[ +L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0 \]

\[ \frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = 0 \]

\[ m \frac{d^2x}{dt^2} + kx = 0 \]

- clamped harmonic oscillator
Solution: $e^{\lambda t}$

\[ L \frac{d^2 e^{\lambda t}}{dt^2} + R \frac{de^{\lambda t}}{dt} + \frac{1}{C} e^{\lambda t} = 0 \]

\[ L \lambda^2 e^{\lambda t} + R \lambda e^{\lambda t} + \frac{1}{C} e^{\lambda t} = 0 \]

\[ L \lambda^2 + R \lambda + \frac{1}{C} = 0 \]

\[ \lambda = -\frac{R}{2} \pm \sqrt{\frac{R^2}{4} - \frac{L}{C}} \]

If $R^2 > \frac{4L}{C}$, \text{ real(\#)} \text{, "over damped"}

\[ \alpha = C_1 e^{-\lambda t} + C_2 e^{R t} \]

\[ \alpha = e^{-\lambda t} \left[ C_1 e^{\sqrt{R^2 - \frac{4L}{C}} t} + C_2 e^{-\sqrt{R^2 - \frac{4L}{C}} t} \right] \]

If $R^2 < \frac{4L}{C}$, imaginary roots $\Rightarrow$ oscillatory solution, "under damped"

\[ \alpha = C_0 e^{-\frac{R}{2} t} \cos(\omega' t + \phi) \]

\[ \omega' = \sqrt{\frac{1}{L C} - \frac{R^2}{4L^2}} \]

If $R^2 = \frac{4L}{C}$, $\alpha = C_1 e^{-\lambda t} + C_2 \lambda e^{\lambda t}$

\[ \text{over damped} \]

\[ \text{critically damped} \]

\[ \text{under damped} \]
Example: At \( t=0 \), a 40-mH inductor is placed in series with a resistance \( R=3 \Omega \) and a charged cap \( C=7.5 \mu F \).

1) Will this circuit oscillate?
   \[
   \text{Is it underdamped? } R^2 < \frac{4L}{C} ?
   \]
   \[
   R^2 = 9 \Omega^2
   \]
   \[
   \frac{4L}{C} = \frac{4(0.04)}{9.8 \times 10^{-6}} = 3.3 \times 10^4 \Omega^2
   \]
   \[
   \checkmark
   \]

2) Determine the frequency
   \[
   f = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4LC}} = \frac{3.6 \times 10^2}{Hz}
   \]

3) What time is the time req. for the charge amplitude to drop to half its steady value?
   \[
   Q = Q_0 e^{-\frac{R}{2L} t} \cos(\omega t + \phi)
   \]
   \[
   \frac{1}{2} Q_0 = Q_0 e^{-\frac{R}{2L} t}
   \]
   \[
   \ln \frac{1}{2} = -\frac{R}{2L} t \quad \Rightarrow \quad t = \frac{2L}{R} \ln 2 \approx 18 \text{ ms}
   \]

4) What value of \( R \) will make the circuit non-oscillatory?
   \[
   \text{Critically damped when}
   \]
   \[
   R^2 = \frac{4L}{C} = 3.3 \times 10^4 \Omega^2
   \]
   \[
   R = 180 \Omega
   \]
   \[
   \Rightarrow \quad R \geq 180 \Omega \text{ for no oscillation}
   \]
AC Source

what happens if the source is oscillating?

\[ I = I_0 \cos 2\pi ft = \pm I_0 \cos \omega t \]

\[ V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \quad I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \]

**Resistor:**

\[ V = IR = I_0 R \cos \omega t = V_{\text{rms}} \cos \omega t \]

Current and voltage "in phase"
in a resistor

\[ P = \overline{IV} = V_{\text{rms}}^2 R = V_{\text{rms}}^2 / R \]

**Inductor:**

\[ V = L \frac{dI}{dt} = 0 \]

\[ V = L \frac{dI}{dt} = L \frac{d}{dt} (I_0 \cos \omega t) = L I_0 (-\omega) \sin \omega t = -\omega LI_0 \sin(\omega t) \quad \sin \theta = -\cos (\theta + 90) \]

\[ V = -\omega LI_0 \cos(\omega t + 90) \]

Voltage and current 90° out of phase

Current lags voltage by 90° in an inductor.
notice $\theta = \pi/2$, average $i \approx 0$

no energy transferred in an inductor on average.

$$\frac{V_o}{I_o} = x_c \quad \text{"inductive reactance"}$$

- max current is less than voltage because inductor
- impedes the flow of charge.

- the greater the frequency or the greater $L$, the
- larger $x_c$ tends to be

also $x_c = \frac{V_{rms}}{I_{rms}}$

* $x_c$ applies only \( \leq \) $\text{at Rms and max}$ values

III Capacitor

$$V = \frac{Q}{C} = 0$$

$$\frac{Q}{C} = \int I_0 \cos \omega t \, dt$$

$$Q = \int I_0 \cos \omega t \, dt$$

$$I_0 \cos \omega t \, dt = \frac{I_0}{\omega} \sin \omega t$$

$$V = \frac{Q^2}{C} = \frac{I_0^2}{\omega^2} \sin \omega t$$

$$\sin \theta = \cos (\pi/2 - \theta) = \cos (\theta - 90°)$$

$$= \sqrt{\frac{I_0}{\omega C}} \cos (\omega t - 90°) = V_0 \cos (\omega t - 90°)$$

peak voltage

the current leads the voltage across a capacitor by $90°$
Similarly define \( X_C \), capacitive reactance:

\[
V_C = I_0 X_C \quad \text{--- Applied only at max or rms values}
\]

\[
X_C = \frac{V_C}{I_0} = \frac{\frac{V_C}{I_0}}{I_0} = \frac{1}{\omega C} = \frac{1}{2\pi f C}
\]

Reactances and resistances are examples of impedance.

Put it all together: LRC series AC circuit

(Who imagined this one up? ---)

\[
V = V_R - V_L + V_C = 0
\]

\[
V = V_R + V_L + V_C
\]

Things are all out of phase... gets messy...

Technique: Phasor Diagram

Consider the unit circle:

\[
\cos \theta = \frac{adj}{hyp} = \frac{x}{1} = x
\]

\[
\sin \theta = \frac{opp}{hyp} = \frac{y}{1} = y
\]

A vector could be rep by unit \( x \) circle.

\[
A \cos \theta = \text{projection of vector onto } x\text{-axis at } \theta
\]
**if \( \theta = wt \) (\( \theta \) changing at some rate \( w \) wrt time)**

Similarly, we can draw arrows for voltage in our circuit!

\[
V_R = V_0 \cos wt = I_0 R \cos wt \\
\text{in phase with current}
\]

at \( t=0 \)

\[
V_{R_0} = I_0 R \\
V_{C_0} = I_0 X_C
\]

Inductor: 90° ahead of current (and \( V_R \))

 Capacitor 90° behind current

After a time \( t \): (rotate whole system)

\[
V_{v_0} = V_0 X_C \\
V_{v_0} = I_0 R \\
V_C = V_{v_0} = I_0 X_C
\]

\( \text{projections on x-axis} \)

\( \text{represent voltages at any} \)

\( \text{gim point in the} \)

what is the voltage across the source?

\[
V = V_R + V_L + V_C
\]
Choose
\[ X_L > X_C \]
\[ \frac{V_L}{I_L} > \frac{V_C}{I_C} \]
\[ V_{L_0} > V_{C_0} \]

\[ V_0 = \sqrt{V_{R_0}^2 + (V_{L_0} - V_{C_0})^2} \]
\[ = \sqrt{(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2} \]
\[ V_0 = I_0 \sqrt{R^2 + (X_L - X_C)^2} \]

Define \( Z \) total impedance
\[ V_L = I_0 Z \]
\[ Z = \sqrt{R^2 + (X_L - X_C)^2} = R^2 + \left( \frac{I_0}{I_0} \right)^2 \]

What is \( \phi \)?

Triangle
\[ \tan \phi = \frac{V_{L_0} - V_{C_0}}{V_{R_0}} = \frac{I_0 (X_C - X_L)}{I_0 R} = \frac{X_L - X_C}{R} \]

Or \( \cos \phi = \frac{V_{R_0} / I_0 Z}{V_0 / I_0 Z} = \frac{R}{Z} \)
What if \( X_L < X_C \)
\[ V_L < V_C. \]

Now current leads the voltage.

For \( X_L < X_C \)

Current leads the voltage by \( \phi \).

For \( X_L > X_C \)

Current lags the voltage by \( \phi \).

Example 20-11