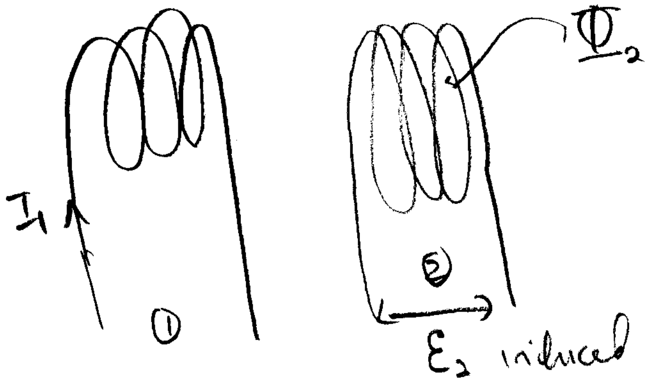
Chapter 30 Inductance, Electromagnetic Oscillations, + AC circuits

Mutual Inductance



flux in ② $\propto I_1$

$N_2 \Phi_{21} \propto I_1$

$$\boxed{\frac{N_2 \Phi_{21}}{I_1} = M_{21}}$$ ← prop. constant, called <u>Mutual Inductance</u>

$\mathcal{E}_2 = -N_2 \frac{d\Phi_{21}}{dt}$ but $\Phi_{21} = \frac{M_{21} I_1}{N_2}$

$= -\cancel{N_2} \frac{d}{dt}\left(\frac{M_{21} I_1}{\cancel{N_2}}\right)$

$$\boxed{\mathcal{E}_2 = -M_{21} \frac{dI_1}{dt}}$$

M_{21} constant in time
→ depends on geometry

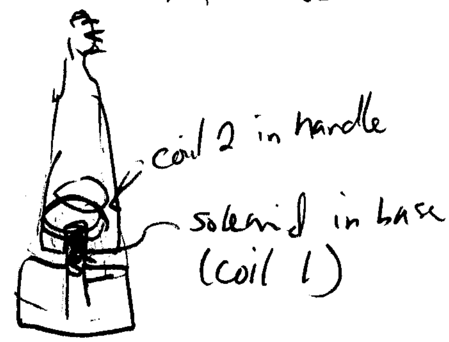
likewise
$\mathcal{E}_1 = -M_{12} \frac{dI_2}{dt}$

it turns out $M_{12} = M_{21}$

so $\boxed{\mathcal{E}_1 = -M\frac{dI_2}{dt} \qquad \mathcal{E}_2 = -M\frac{dI_1}{dt}}$

unit for M is henry H $\quad 1\,H = 1\,V\cdot s/A = 1\,\Omega\cdot s$

Example: electric tooth brush charger



given: base is solenoid of length ℓ, N_B turns, and area A, + current I. The handle has N_H turns. find the mutual inductance

$$M = \frac{N_H \Phi_{BH}}{I} = \frac{N_H BA}{I} \qquad B = \mu_0 \frac{N_B I}{\ell}$$

$$M = \mu_0 \frac{N_B N_H A}{\ell}$$

Self-Inductance

As current increases B created by current increases and so does flux $(\vec{B}\cdot\vec{A})$. This induces an emf which resists the increasing current.

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

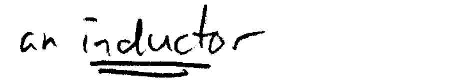
define $L = \frac{N\Phi_B}{I} \rightarrow \Phi_B = \frac{LI}{N}$

self inductance

$$\mathcal{E} = -\cancel{N}\frac{d}{dt}\left(\frac{LI}{\cancel{N}}\right)$$

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

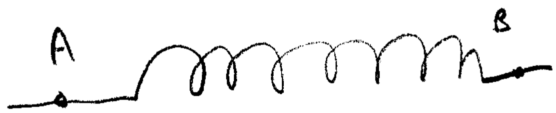
A coil in a circuit is called an inductor



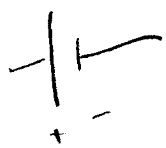
a) I → increasing

which way is emf?

b) I → decreasing

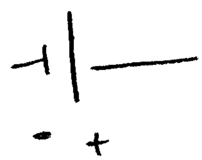


a) like



makes current flow in opposite direction

b) like



eg tesla coil

do demo

Energy Stored in a Magnetic field

$$P = IV \Rightarrow P = I\mathcal{E} = I L \frac{dI}{dt}$$

$$dW = P\,dt = LI \cancel{dt} \frac{dI}{\cancel{dt}} = LI\,dI$$

$$W = \int dW = \int LI\,dI = \frac{LI^2}{2}$$

work done = energy stored

$$\boxed{U = \frac{1}{2} L I^2}$$

→ similar to cap $U = \frac{1}{2} C V^2$

in terms of magnetic field $B = \mu_0 I n = \mu_0 \frac{IN}{l} \rightarrow I = \frac{Bl}{\mu_0 N}$

$$L = \frac{N \Phi_B}{I} = \frac{N \mu_0 n \cancel{I} A}{\cancel{I}} = \frac{N^2 \mu_0 A}{l}$$

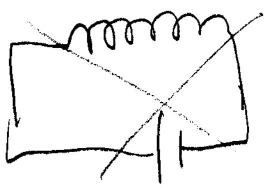
$$U = \frac{1}{2}\left(\frac{\cancel{N^2} \mu_0 A}{l}\right)\left(\frac{Bl}{\mu_0 \cancel{N}}\right)^2 = \frac{1}{2}\frac{B^2}{\mu_0} A l$$

volume!

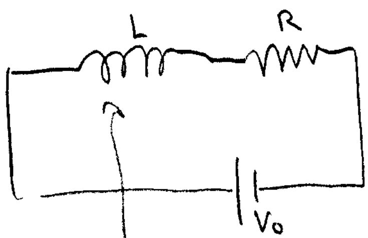
energy density

$$\boxed{u = \frac{1}{2}\frac{B^2}{\mu_0}}$$

Induction in Circuits



impossible - wires have resistance



"LR circuit"

recall $\mathcal{E} = -L\frac{dI}{dt}$

"Charging"

Loop rule: $V_0 - L\frac{dI}{dt} - IR = 0$

$$L\frac{dI}{dt} + IR = V_0$$

separate variables:

$$L\,dI = (V_0 - IR)\,dt$$

$$\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_0^t \frac{dt}{L}$$

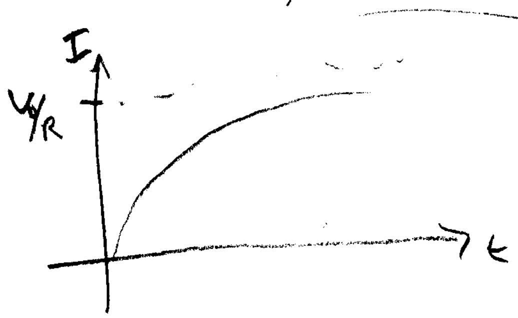
$$-\frac{1}{R}\ln(V_0 - IR)\Big|_0^I = \frac{t}{L}$$

$$\ln\left(\frac{V_0 - IR}{V_0}\right) = -\frac{t}{L}R$$

$$1 - \frac{IR}{V_0} = \frac{V_0 - IR}{V_0} = e^{-t\frac{R}{L}}$$

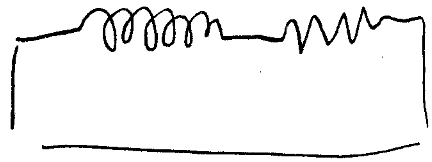
time constant

$$\boxed{I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right)} \quad \text{where} \quad \boxed{\tau = \frac{L}{R}}$$



LRC-2

take out battery: at $t=0$



$$\cancel{V_0} - IR - L\frac{dI}{dt} = 0$$

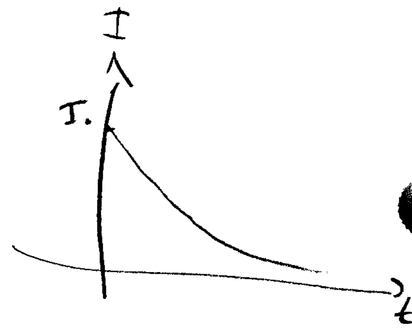
$$-IR = L\frac{dI}{dt}$$

$$-\int_0^t \frac{R}{L}\,dt = \int_{I_0}^{I} \frac{1}{I}\,dI$$

$$-\frac{R}{L}t = \ln\left(\frac{I}{I_0}\right)$$

$$e^{-t\frac{R}{L}} = I/I_0$$

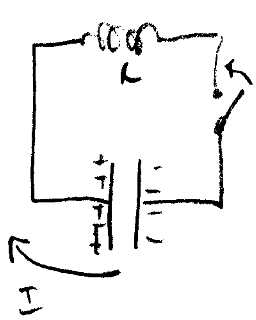
$$\boxed{I = I_0 e^{-t/\tau} \quad \tau = L/R}$$



Add capacitors LRC circuits.

- first: consider idealized circuit - LC:

Charged capacitor + inductor =



switch closed @ $t=0$

loop: $-L\frac{dI}{dt} + \frac{Q}{C} = 0$

$I = -\frac{dQ}{dt}$ charge leaving capacitor is creating the current

$$-L\frac{d}{dt}\left(-\frac{dQ}{dt}\right)+\frac{Q}{C}=0$$ divide by L

$$\frac{d^2Q}{dt^2}+\frac{Q}{CL}=0$$ 2nd order diffy Q

$$\frac{d^2Q}{dt^2}=-\frac{1}{CL}Q$$

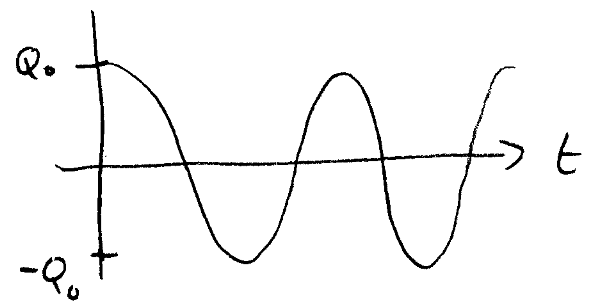
$$\Rightarrow \quad Q = Q_0\cos(\omega t+\phi)$$

check: $\frac{dQ}{dt} = Q_0\,\omega(-\sin(\omega t+\phi))$

$$\frac{dQ^2}{dt^2} = -Q_0\,\omega^2\underbrace{\cos(\omega t+\phi)}_{Q}$$

$$= -\omega^2 Q$$

so $\frac{1}{LC}=\omega^2 \Rightarrow \omega = \sqrt{\frac{1}{LC}}$ ← frequency of the circuit



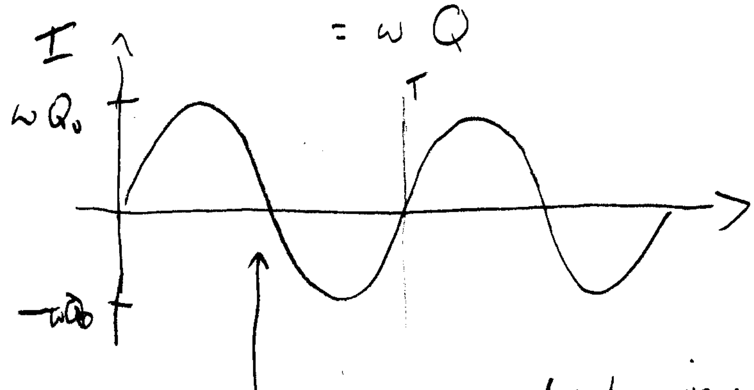
charge on the capacitor oscillating

what about I?

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\left(Q_0\cos(\omega t+\phi)\right)$$

$$= \omega Q_0 \sin(\omega t+\phi)$$

$$= \omega Q$$



current in the inductor

current reverses - discharging, charging, etc. ... on and on

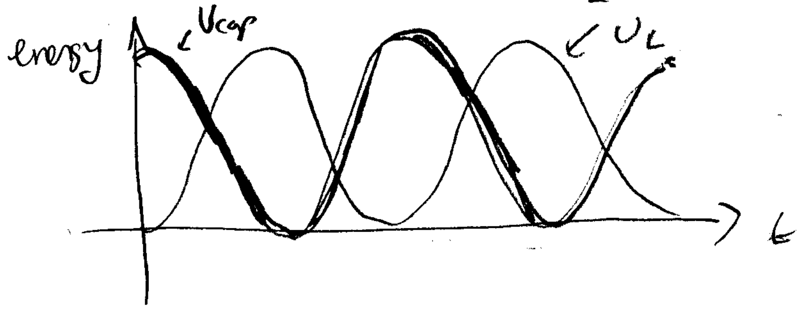
LRC-4

what's happening to the energy?

$$U_{cap} = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\omega t+\phi)$$

$\omega^2 = \frac{1}{LC}$

$$U_{inductor} = \frac{1}{2}LI^2 = \frac{L\omega^2 Q_0^2}{2}\sin^2(\omega t+\phi) = \frac{Q_0^2}{2C}\sin^2(\omega t+\phi)$$



total energy

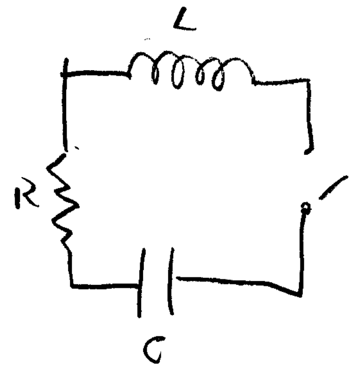
$$U_{total} = \frac{Q_0^2}{2C}\cos^2(\omega t+\phi) + \frac{Q_0^2}{2C}\sin^2(\omega t+\phi)$$

$$= \frac{Q_0^2}{2C}\left[\underbrace{\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)}_{=1}\right]$$

$$= \frac{Q_0^2}{2C} \leftarrow \text{constant!}$$

✓ energy conserved

Real life: LRC → guess what will happen?



loop:

$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0$$

$$I = -\frac{dQ}{dt}$$

$$+L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \leftarrow \text{damped harmonic oscillator}$$

Solution $e^{\lambda t}$

$$L\frac{d^2 e^{\lambda t}}{dt^2} + \frac{R}{\cancel{L}}\frac{d}{dt}e^{\lambda t} + \frac{1}{\cancel{L}C}e^{\lambda t} = 0$$

$$L\lambda^2 \cancel{e^{\lambda t}} + \frac{R}{\cancel{L}}\lambda \cancel{e^{\lambda t}} + \frac{1}{\cancel{L}C}e^{\lambda t} = 0$$

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0$$

$$\lambda = -R \pm \sqrt{R^2 - 4\frac{L}{C}}$$

if $R^2 > 4\frac{L}{C}$ ↑ real #, "over damped"

$$Q = C_1 e^{(-R+\sqrt{R^2-\frac{4L}{C}})t} + C_2 e^{(-R-\sqrt{R^2-\frac{4L}{C}})}$$

$$= e^{-Rt}\left[C_1 e^{\sqrt{R^2-\frac{4L}{C}}\,t} + C_2 e^{-\sqrt{R^2+\frac{4L}{C}}\,t}\right]$$

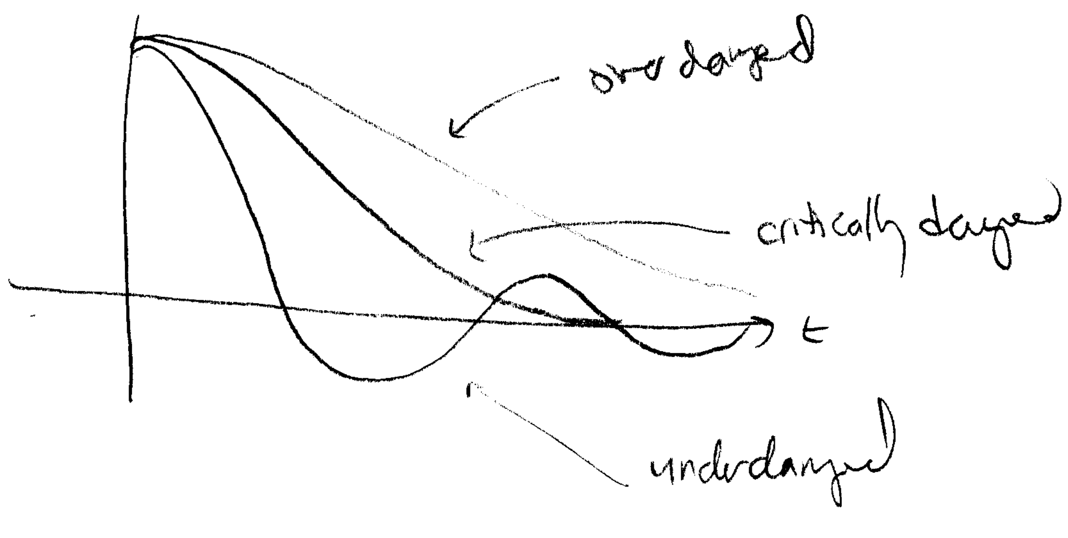
if $R^2 < \frac{4L}{C}$ imaginary roots ⇒ oscillating solution

"under damped"

$$Q = Q_0 e^{-\frac{R}{2L}t}\cos(\omega' t + \phi)$$

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

if $R^2 = \frac{4L}{C}$ $\quad Q = C_1 e^{\lambda t} + C_2 \lambda e^{\lambda t}$



Example: At $t=0$, a 40-mH inductor is placed in series w/ a resistance $R = 3\,\Omega$ and a charged cap $C = 4.5\,\mu F$.

(a) will this circuit oscillate?

sol is it underdamped? $R^2 < \frac{4L}{C}$?

$$R^2 = 9\,\Omega^2$$

$$\frac{4L}{C} = \frac{4(0.04)}{4.8 \times 10^{-6} F} = 3.3 \times 10^4\,\Omega^2$$

✓

b) Determine the frequency

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 3.6 \times 10^2\ Hz$$

c) What ~~time~~ is the time req. for the charge amplitude to drop to half its starting value?

$$Q = \underbrace{Q_0 e^{-\frac{R}{2L}t}}_{\text{amplitude}} \cos(\omega' t + \phi)$$

$$\frac{1}{2}\cancel{Q_0} = \cancel{Q_0} e^{-R/2L\,t}$$

$$\ln\frac{1}{2} = -\frac{R}{2L}t \rightarrow t = \frac{2L}{R}\ln 2 = 18\,ms$$

d) what value of R will make the circuit non-oscillatory?

critically damped when

$$R^2 = \frac{4L}{C} = 3.3 \times 10^4\,\Omega^2$$

$$R = 180\,\Omega$$

so $R \geq 180\,\Omega$ for no oscillation

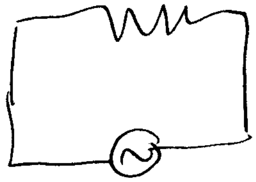
AC Source

what happens if the source is oscillating?

$I = I_0 \cos 2\pi f t = I_0 \cos \omega t$

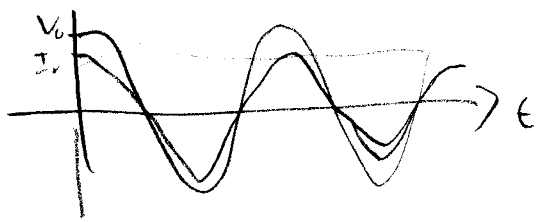
$\left(V_{rms} = \frac{V_0}{\sqrt{2}} \quad I_{rms} = \frac{I_0}{\sqrt{2}} \right)$

I resistor:



$V = IR = I_0 R \cos \omega t$

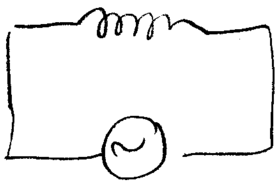
$= V_0 \cos \omega t$



current and voltage "in phase" in a resistor

$\bar{P} = \bar{I}\bar{V} = I_{rms}^2 R = V_{rms}^2 / R$

II inductor



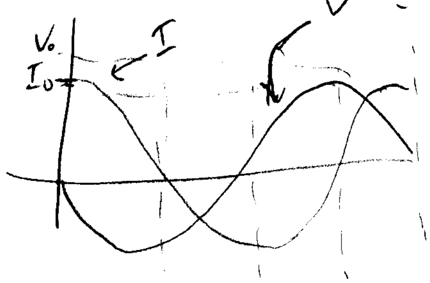
loop rule $V - L\frac{dI}{dt} = 0$

$V = L\frac{dI}{dt} = L\frac{d}{dt}\left(I_0 \cos \omega t \right)$

$= L I_0 (-\omega) \sin \omega t$

$= -\omega L I_0 \sin(\omega t)$ ← $\sin\theta = -\cos(\theta + 90)$

$V = \omega L I_0 \cos(\omega t + 90)$



voltage + current $90°$ out of phase

current lags voltage by $90°$ in an inductor notice $P = IV$ average is ~ 0

no energy transferred in a inductor on average

like $V = IR$

(reactance a bit like resistance)

consider $\frac{V_0}{I_0} = X_L$ "inductive reactance"

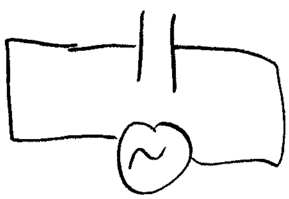
max current is less than max voltage because inductor impedes the flow of charge.

- the greater the frequency and the greater L, the larger X_L tends to be

also $X_L = \frac{V_{rms}}{I_{rms}}$

* X_L applies only at Rms and max values

III Capacitor



$V = \frac{Q}{C} = 0$

$V = \frac{Q}{C}$

$Q = ?$

$I = I_0 \cos \omega t = \frac{dQ}{dt}$

$I \cos \omega t \, dt = dQ$

$Q = \int I_0 \cos \omega t \, dt$

$= \frac{I_0}{\omega} \sin \omega t$

$V = \frac{Q}{C} = \frac{I_0}{C\omega} \sin \omega t$

$\sin \theta = \cos(90^\circ - \theta)$
$= \cos(\theta - 90^\circ)$

$= \frac{I_0}{\omega C} \cos(\omega t - 90) = V_0 \cos(\omega t - 90)$

peak voltage

the current leads the voltage across a capacitor by 90°

Similarly define X_C capacitive reactance

$V_0 = I_0 X_C$ ← *Applies only at max or rms values

$$X_C = \frac{V_0}{I_0} = \frac{\frac{I_0}{\omega C}}{I_0} = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

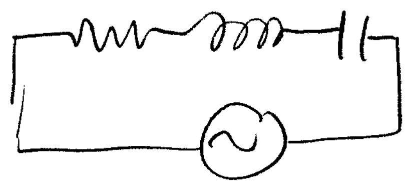
reactances and resistances are examples of <u>impedance</u>

<u>Put it all together</u>: LRC series AC circuit

(who imagined this one up? ...)

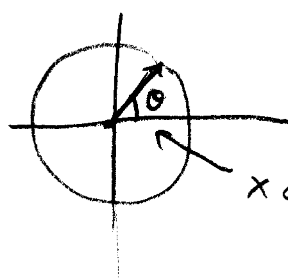
$V - V_R - V_L - V_C = 0$

$V = V_R + V_L + V_C$



things are all out of phase...
gets messy ...
technique: <u>Phasor Diagram</u>

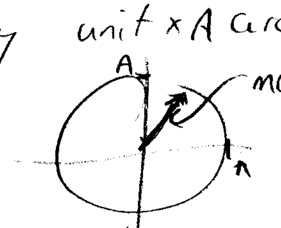
consider the unit circle:



$$\cos\theta = \frac{adj}{hyp} = \frac{x}{1} = x$$

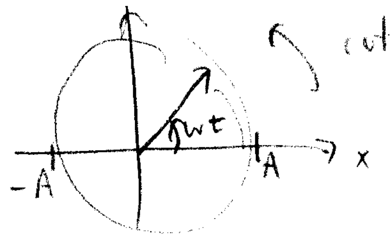
$$\sin\theta = \frac{opp}{hyp} = \frac{y}{1} = y$$

$A\cos\theta$ could be rep by unit × A circle:



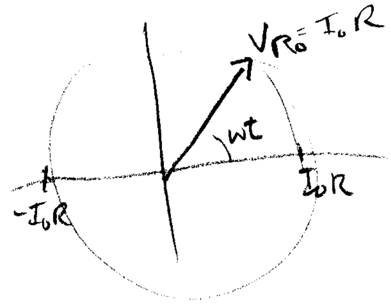
$A\cos\theta$ = projection of vector onto x-axis at any θ

if $\theta = \omega t$ (θ changing at some rate ω wrt time)



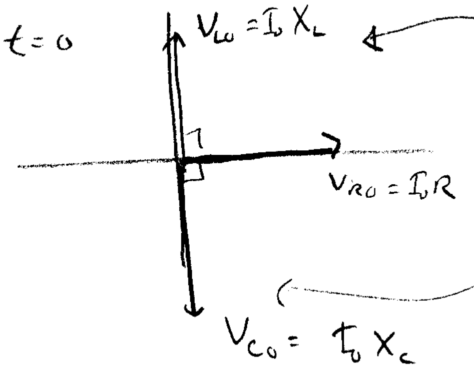
rotating with time

Similarly we can draw arrows for voltage in our circuit!



$V_R = V_0 \cos\omega t = I_0 R \cos\omega t$
(in phase w/ current)

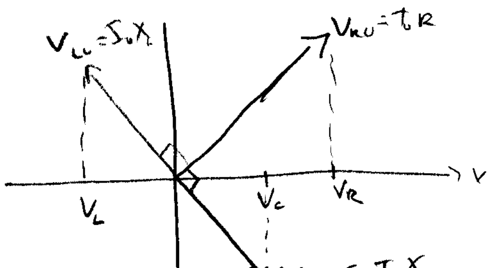
at $t=0$



Inductor: 90° ahead of current (and V_R)

Capacitor 90° behind current

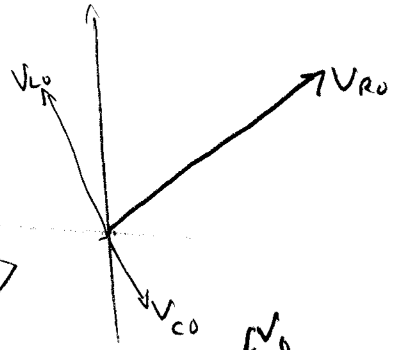
After a time t: (rotate whole system)



← projections on x-axis represent voltages at any given point in time

what is the voltage across the source?

$V = V_R + V_L + V_C$



~~V_0 ...~~

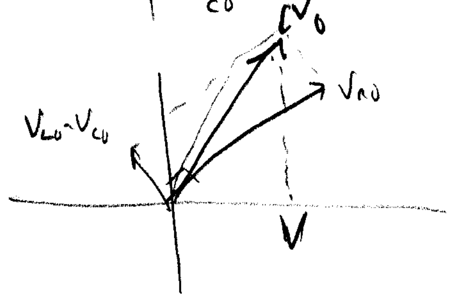
add vectors

Choose

$X_L > X_C$

$\frac{V_{L0}}{I_{L0}} > \frac{V_{C0}}{I_{C0}}$

$V_{L0} > V_{C0}$



$V = V_0 \cos(\omega t + \phi)$

↑ some angle, by which it is out of phase with the current

$$V_0 = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2}$$

$$= \sqrt{(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2}$$

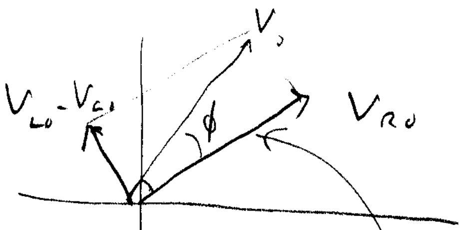
$$V_0 = I_0 \underbrace{\sqrt{R^2 + (X_L - X_C)^2}}$$

define Z total impedence

$$V_0 = I_0 Z$$

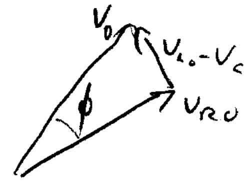
$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

what is ϕ ?



V_{R0} is in phase w/ current

triangle

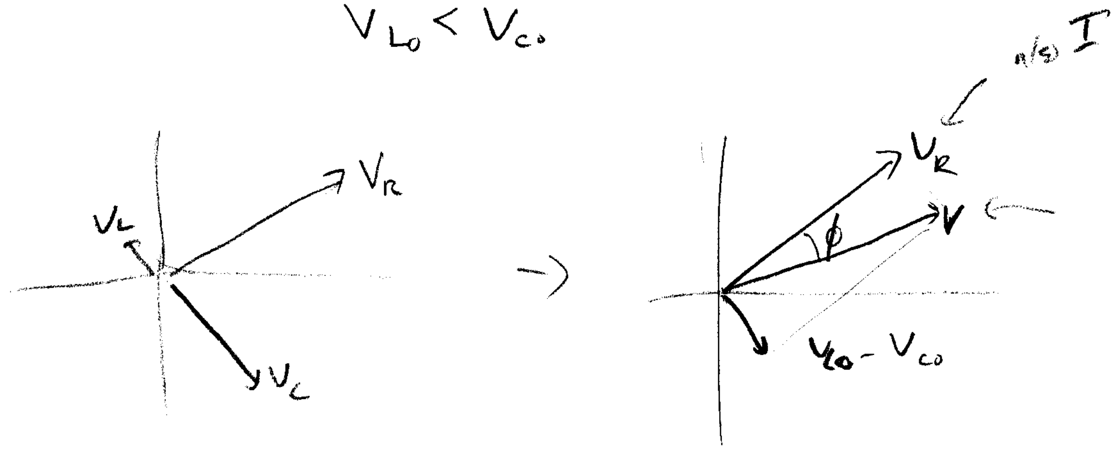


$$\tan\phi = \frac{V_{L0} - V_{C0}}{V_{R0}} = \frac{I_0 (X_L - X_C)}{I_0 R} = \frac{X_L - X_C}{R}$$

$$\text{or } \cos\phi = \frac{V_{R0}}{V_0} = \frac{I_0 R}{I_0 Z} = \frac{R}{Z}$$

what if $X_{L0} < X_{C0}$

$V_{L0} < V_{C0}$



now current leads the voltage

for $X_L < X_C$
current leads the voltage by ϕ

for $X_L > X_C$
current lags the voltage by ϕ

Example 30-11