III Electric Flux + Gauss's Law

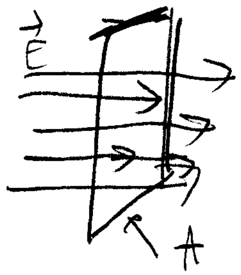
↳ 1777–1855

now we know the theory, we learn important tricks to avoid messy integrals

A) Flux

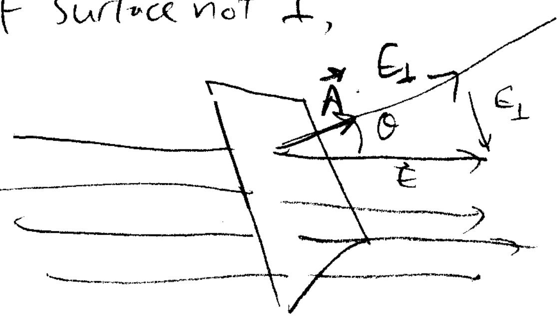
~~def~~: flux is a measure of the electric field passing through a given area.

Since # of field lines is prop to the strength of the field, we can think of the flux as the number of field lines going through an area perpendicular to the field lines:



$$\text{Flux} = \Phi_E = |E||A|$$

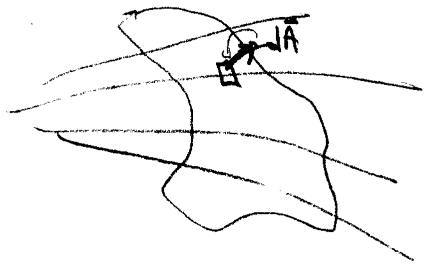
if surface not $\perp$,



$$\boxed{\begin{aligned}\Phi_E &= |E||A| \cos\theta \\ \Phi &= \vec{E}\cdot\vec{A}\end{aligned}}$$

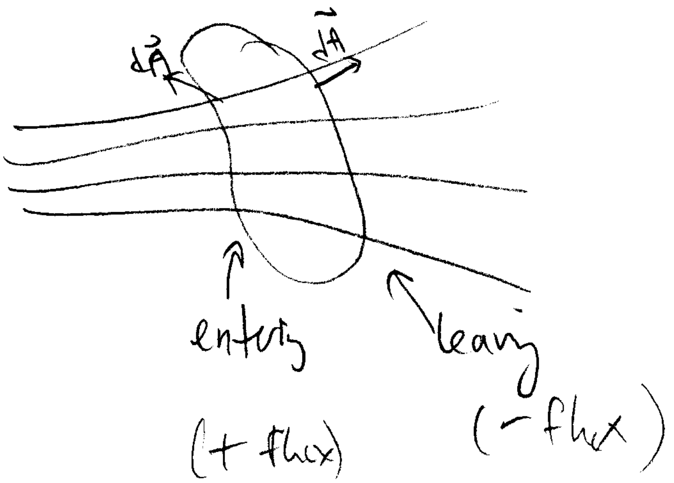
for strange shapes, we need to integrate:

$$\Phi_E = \int \vec{E}\cdot d\vec{A}$$



flux E

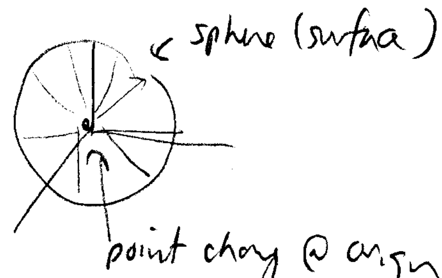
What about a closed surface?



$$\Phi_E = \Phi_{in} + \Phi_{out} = 0$$

Gauss' Law

Consider:



$$\int \vec{E}\cdot d\vec{a} = \int |E||da|\cos 0$$

$$= \int \frac{kq}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{kq}{r^2}\int da$$

$$= \frac{kq}{\cancel{r^2}} 4\pi \cancel{r^2} \qquad \text{but } k = \frac{1}{4\pi\epsilon_0}$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0} q \cancel{4\pi}$$

$$= \frac{q}{\epsilon_0}$$

notice r canceled! So could have been any enclosed surface — not even a sphere, + would have ~~trapped~~ had same # of field lines $\ni$ it!

now generalize!

let there be multiple charges:

$$\int \vec{E} \cdot d\vec{a} = \int (\vec{E}_1 + \vec{E}_2 + \cdots) \cdot d\vec{a}$$

$$= \int (\vec{E}_1 \cdot d\vec{a} + \vec{E}_2 \cdot d\vec{a} + \cdots)$$

$$= \sum_{i=1}^{n} \left(\oint \vec{E}_i \cdot d\vec{a} \right)$$

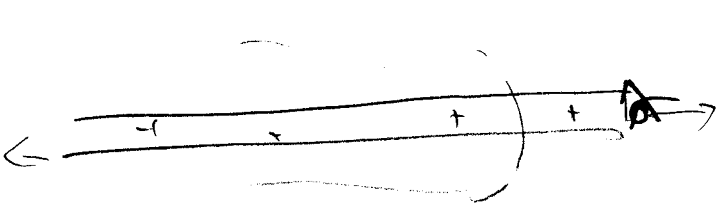
$$= \sum_{i=1}^{n} \left(\frac{q_i}{\epsilon_0} \right) \qquad \text{let } Q = \sum_{i=1}^{n} q_i$$

$$= \frac{Q}{\epsilon_0}$$

So for any closed surface,

$$\boxed{\int \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0}}$$ Gauss's Law

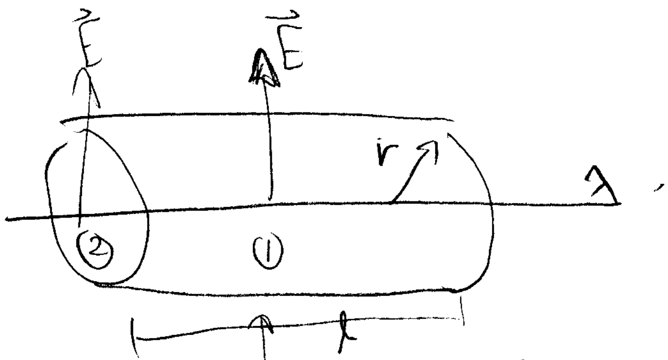
<u>Example</u>: revisit long line of charge:



notice symmetry which ~~(check later)~~ way must E point?

pick gaussian surface so that the angle between $\vec{E}$ and $d\vec{a}$ is always either $0°$ or $90°$

flux 4



① side surface $\vec{E} \parallel \vec{A}$

so $\int \vec{E} \cdot d\vec{A} = E\, 2\pi r l$

② ends $\vec{E} \perp \vec{A}$

so $\int \vec{E} \cdot d\vec{A} = 0$

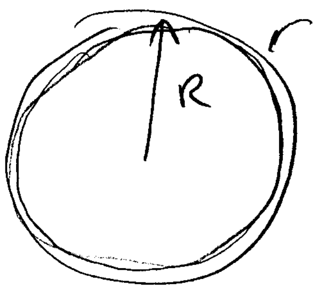
so $\int \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$ ∴ Gauss' law

$E\, 2\pi r \cancel{l} = \frac{\lambda \cancel{l}}{\epsilon_0}$

$E = \frac{\lambda}{2\pi\epsilon_0 r}$ check $E = 2k\frac{\lambda}{r}$ ✓

$\ $ x

Class Practice w/ another classic problem: (ex 22-3)
Spherical conductor



Q uniformly distributed on shell

(a) find $\vec{E}$ everywhere outside sphere
(b) " " " inside sphere
(c) what if it were a solid spherical conductor?

(also do ex 22-4, 5, at least as HW or workshop)

Solution

Inside $Q_{enc} = 0 \Rightarrow E = 0$

outside $Q_{encl} = Q$

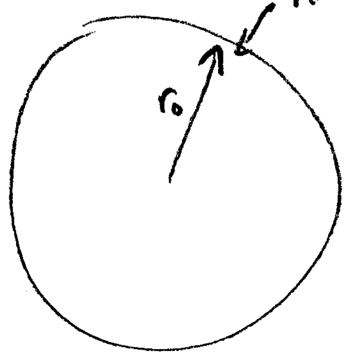
$$\oint \vec{E} \cdot d\vec{A} = E\, 4\pi r^2$$

$$E\, 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}}$$

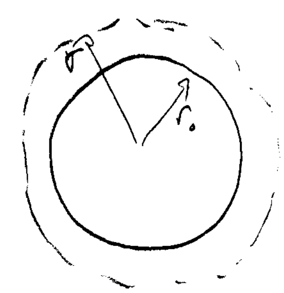
note: gauss' law gives magnitude, while we must figure out direction

Example: Solid sphere of charge

noncondcting sphere, charge Q



find E inside ($r < r_0$) and outside ($r > r_0$)



outside:

$$Q_{enc} = Q$$

$$\int \vec{E} \cdot d\vec{A} = E\, 4\pi r^2$$

$$\text{so } E\, 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow E_{out} = \frac{Q}{4\pi r^2 \epsilon_0}$$

notice looks like pt charge

inside know Q, what is ρ?

$$\rho = \frac{\text{charge}}{\text{volume}} = \frac{Q}{\frac{4}{3}\pi r_0^3}$$

$$Q_{enc} = \int_0^r \rho\, dV = \frac{Q}{\frac{4}{3}\pi r_0^3} \int dV$$

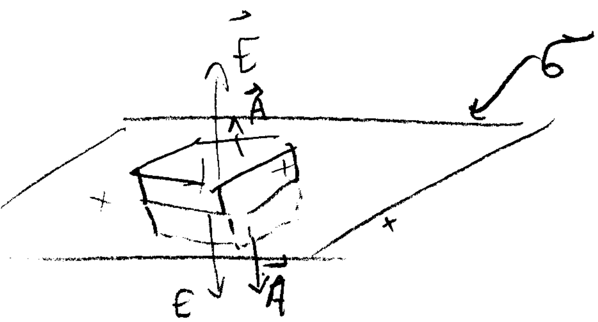
$$Q_{enc} = \frac{Q\, \frac{4}{3}\pi r^2}{\frac{4}{3}\pi r_0^3} = \frac{Q r^3}{r_0^3}$$

$$\text{so } E\, 4\pi r^2 = \frac{Q}{\epsilon_0} \frac{r^3}{r_0^3}$$

$$E = \frac{Q}{\epsilon_0} \frac{r^{\cancel{3}}}{4\pi \cancel{r^2}\, r_0^3} = \frac{Q\, r}{4\pi \epsilon_0 r_0^3}$$

$\longleftarrow$ increases w/ increasing $r \rightarrow$ makes sense

Example - Infinite plane of charge



sides : $\vec{E}\cdot\vec{A} = 0$ bcs $\theta = 90^\circ$

Top: $\theta = 0$ $A = A$:)

bottm $\theta = 0$

$$\int \vec{E}\cdot\vec{A} = \frac{Q}{\epsilon_0}$$

$$\underbrace{E\cancel{A}}_{top} + \underbrace{E\cancel{A}}_{bottm} = \frac{\sigma \cancel{A}}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$