Rather than try to "construct" viscous transport from first principles as attempted (and done very incorrectly in some textbooks) let's assume that turbulence acts as a viscosity to then derive the accretion disk transport equations. Note that this "assumption" is really equivalent to what is currently used in disk modeling for direct comparisons with observations but not a fundamentally consistent or complete approach. It is a theoretical frontier to improve the theory.

So for the present we will explicitly assume a closure for which the Reynolds stress terms associated with turbulent fluctuations in Navier-Stokes equation take the form:

\[ \overline{u'' \cdot u'''} = -\nabla x (\overline{u' \cdot \nabla x u}) \]  

where \( \overline{\cdot} \) is the fluctuation, \( \overline{\cdot} \) mean, and

\[ \nu_T = \frac{\overline{\nu}}{2H} \] = turbulent viscosity

inviscid viscosity.

See next page for \( \overline{\nu} \) and \( H \).
The continuity equation is given by

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{for } \rho = \rho(r, \phi, z), \quad \mathbf{u} = \mathbf{u}(r, \phi, z) \]  \hspace{1cm} (2r)

Define

\[ \bar{\rho} = \frac{\int_{0}^{2\pi} \int_{-H}^{H} \rho \, d\phi \, dz}{2\pi H} \quad \text{mean surface density} \]

\[ \bar{\mathbf{u}} = \frac{\int_{0}^{2\pi} \int_{-H}^{H} \rho \mathbf{u} \, d\phi \, dz}{\int_{0}^{2\pi} \int_{-H}^{H} \rho \, d\phi \, dz} \]

\[ \bar{\mathbf{u}} = \frac{1}{2\pi \bar{\rho}} \int_{0}^{2\pi} \int_{-H}^{H} \rho \mathbf{u} \, d\phi \, dz \]

\[ \bar{\rho} \bar{\mathbf{u}} = \frac{1}{2\pi \bar{\rho}} \int_{0}^{2\pi} \int_{-H}^{H} \rho \mathbf{u} \, d\phi \, dz \quad \text{density weighted mean velocity} \]

\[ \Rightarrow \bar{\rho} = \bar{\rho}(r), \quad \bar{\mathbf{u}} = \bar{\mathbf{u}}(r) \quad (\text{\(z, \phi\) are averaged out}) \]

Then, after integrating over \( d\phi \, dz \) \( (1r) \rightarrow \)

\[ \frac{\partial \bar{\rho}}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \bar{\rho} \bar{\mathbf{u}}_R \right) = 0 \quad \text{(cylindrical coords)} \]  \hspace{1cm} (3r)

Similarly, from the \( \phi \) component of Navier-Stokes:

\[ \bar{\mathbf{u}}_\phi \left( \frac{\partial \bar{\mathbf{u}}_\phi}{\partial t} + \bar{\mathbf{u}}_R \frac{\partial \bar{\mathbf{u}}_\phi}{\partial R} + \bar{\mathbf{u}}_R \bar{\mathbf{u}}_\phi - \frac{\bar{\mathbf{u}}_R \bar{\mathbf{u}}_R}{R^2} \right) = \frac{\partial}{\partial R} \left( \bar{\rho} \frac{\partial \bar{\mathbf{u}}_\phi}{\partial R} \right) + \frac{1}{2} \frac{\partial}{\partial R} \left( \frac{\bar{\rho} \bar{\mathbf{u}}_\phi \bar{\mathbf{u}}_\phi}{R^2} \right) - \frac{\bar{\rho} \bar{\mathbf{u}}_\phi}{R^2} - \frac{\bar{\rho} \bar{\mathbf{u}}_\phi}{R} \]  \hspace{1cm} (4r)
Hereafter for notational simplicity
I drop the overbars on \( \bar{u}, \bar{\zeta} \) and write
\( \bar{V} = V \). That is \( \bar{u} \to u \) and \( \bar{\zeta} \to \zeta \).
Then multiply eqn (3v) by \( RU\phi \):

\[
RU\phi \frac{\partial \bar{\zeta}}{\partial t} + u\phi \frac{\partial}{\partial R} (RU\phi) = 0 \quad (5r)
\]

and multiply eqn (3v) by \( R^2 \):

\[
\Rightarrow
\]

\[
RU\phi \frac{\partial \bar{\zeta}}{\partial t} + RU\phi \frac{\partial \bar{\zeta}}{\partial R} + \Sigma UR\phi u\phi
\]

\[
= R \frac{\partial}{\partial R} \left( RU \frac{\partial \bar{\zeta}}{\partial R} \right) + \frac{\partial (R \Sigma UR\phi)}{\partial R} - \frac{\Sigma UR\phi}{R} \frac{\partial u\phi}{\partial R} \quad (6r)
\]

Footnote: The \( \phi \) component of the axisymmetric Navier-Stokes equation
equation that arises if one assumes \( \bar{\zeta} = 0 \) of all quantities and assumes \( u = \bar{u} \)
and \( \phi = \bar{\phi} \) and simply replaces \( R \bar{\zeta} \) with \( \bar{\zeta} \) is

\[
\bar{\phi} \left( \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial R} + \bar{\phi} \frac{\partial \bar{u}}{\partial R} \right) = \frac{3}{2R} \left( \frac{\partial (\rho \bar{\phi} \bar{u} \bar{u})}{\partial R} \right) + \frac{1}{R} \frac{\partial}{\partial R} \left( \rho \bar{u} \bar{u} \partial \bar{u} \partial \bar{u} \right) + \frac{1}{R^2} \partial \varphi \partial \bar{\bar{u}} \partial \bar{\bar{u}} - \frac{\bar{\bar{u}} \bar{\bar{u}}}{R^2} \frac{\partial \bar{\bar{u}}}{\partial R}
\]

Eqn (6r) can be derived by by integrating this over \( \bar{\varphi} \). Often
the distinction between \( u \) and \( \bar{u} \) is incorrectly ignored so one should really formally average.
Add (5r) + (6r) using \( \mathcal{N} = \frac{\mathcal{U} \varphi}{\mathcal{R}} \).

\[
\frac{\partial}{\partial t} \mathcal{E} (\mathcal{Z} \mathcal{U} \varphi) + \frac{\partial}{\partial \mathcal{R}} \left( \mathcal{R} \mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi) \right) + \mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi) = \frac{\mathcal{R}}{\mathcal{Z} \mathcal{R}} \left( \mathcal{V} \mathcal{E} (\mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi)) \right) + \mathcal{E} \left( \frac{\mathcal{V} \mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi)}{\mathcal{Z} \mathcal{R}} \right) - \mathcal{V} \mathcal{E} (\mathcal{Z} \mathcal{U} \varphi) - \mathcal{E} \mathcal{V} \mathcal{E} (\mathcal{Z} \mathcal{U} \varphi) - \frac{\mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi)}{\mathcal{Z} \mathcal{R}}
\]

\[
\frac{\partial}{\partial t} \mathcal{E} (\mathcal{Z} \mathcal{U} \varphi) + \frac{1}{2 \mathcal{R}} \left( \mathcal{R} \mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi) \right) = \frac{1}{\mathcal{R}} \mathcal{E} \left( \frac{\mathcal{V} \mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi)}{\mathcal{Z} \mathcal{R}} \right) + \mathcal{E} \left( \frac{\mathcal{V} \mathcal{E} (\mathcal{Z} \mathcal{U} \mathcal{R} \varphi)}{\mathcal{Z} \mathcal{R}} \right)
\]
\[ \frac{d}{dr} \left( \frac{R_2^2 - R_1^2}{2} \right) = \frac{F}{2 \pi R^2} \]

\[ T = F \left( \frac{R_2}{2 \pi} \right) \]

\[ \Delta \theta = \frac{\tau}{I} \]

\[ \tau = \frac{R}{2 \pi} \left( \frac{1}{2} \frac{d^2}{dr^2} \right) \]

\[ \phi = \frac{1}{2} R^2 \theta \]

\[ \frac{d\phi}{dt} = \frac{1}{2} \frac{d^2 \theta}{dt^2} \]

\[ \frac{d^2 \theta}{dt^2} = \frac{1}{I} \left( F - \frac{\tau}{R} \right) \]

\[ \frac{d^2 \theta}{dt^2} = \frac{1}{I} \left( F - \frac{\tau}{R} \right) \]

\[ \frac{d^2 \theta}{dt^2} = \frac{1}{I} \left( F - \frac{\tau}{R} \right) \]

\[ \frac{d^2 \theta}{dt^2} = \frac{1}{I} \left( F - \frac{\tau}{R} \right) \]
Check physical consistency:

\[ G = 0 \text{ for } \frac{dG}{dR} = 0 \quad \checkmark \]

\[ G < 0 \text{ for } \frac{dG}{dR} < 0 \quad \checkmark \]

\[ \text{total torque on ring of gas between } R, R + dR: \]

\[ G(R + dR) - G(R) = \frac{2G}{dR} \quad \text{dR} = dG. \quad \text{Now} \]

rate of work = \( d\vec{F} \cdot \vec{v} = d\vec{F} \cdot (\vec{F} \times d\vec{r}) \)

\[ = \vec{r} \cdot (\vec{F} \times d\vec{r}) \]

\[ = \vec{r} \cdot d\vec{F} = \pm 2\pi R \delta \chi \]

\[ \Rightarrow \text{rate of work} \quad (\text{because } d\vec{G} \parallel \pm \vec{r}) \]

\[ = \int_{2R} 2\pi R d\chi = 2(\pi R) \frac{dR}{dR} = G \frac{\partial G}{\partial R} dR \]

integrate: \[ \Rightarrow \text{total work rate} \]

\[ = \int_{\text{boundary}} \frac{2(\pi R)}{dR} dR - \int_{\text{internal dissipation}} \frac{G \frac{\partial G}{\partial R}}{dR} dR \]
dissipation term converts mechanical energy into particle energy \( \rightarrow \) heat \( \rightarrow \) radiation

per area (2 faces of ring) \( \Rightarrow \)

\[
\frac{c \frac{dN}{dR}}{4 \pi R^2} = \frac{c \left( R^2 \right) \frac{dN}{dR}}{4 \pi R^2}
\]

\( = + \frac{1}{2} \sum R^2 \left( \frac{dN}{dR} \right)^2 \) (from (10c) (p. 143))

\[ D(R) = \text{energy loss rate per unit area from dissipation} \]

note we need to have \( \frac{dN}{dR} \neq 0 \)

need to know \( \Delta, \Sigma \) to compare to observations.
Viscosity can be estimated by characteristic velocity and length scale associated with particle motions & deflections.

The force density associated with the viscosity of the previous section comes from the $\mathbf{f} = \nabla \mathbf{V}$ term in Navier-Stokes equation. Recall that $\mathbf{V} = \mathbf{V}_T + \mathbf{V}_{\text{microphys}}$ with $\mathbf{V}_T \gg \mathbf{V}_{\text{microphys}}$ to recall its importance. We can compute the Reynolds number: ratio of $\mathbf{V} \cdot \nabla \mathbf{V}$ term to $\nabla^2 \mathbf{V}$ term small for $\mathbf{V} \approx \mathbf{V}_0$, $D_T = \frac{1}{R}$, $\mathbf{V} \approx \ell \mathbf{V}_T + \ell \mathbf{V}_{\text{microphys}}$

$$\Rightarrow \quad \frac{\mathbf{V} \cdot \nabla \mathbf{V}}{\nabla^2 \mathbf{V}} \approx \frac{R \mathbf{V}_0}{\ell \mathbf{V}_T} = Re_{,\text{eff}} \ll 1$$

Note: If turbulence were absent, recall that $\ell = \text{microphysical deflection scale}$ from Coulomb collisions for protons

$C_s = \text{average proton speed}$

$$Re_{,\text{micro}} = \frac{R \mathbf{V}_0}{\ell \mathbf{V}_{\text{micro}}} \approx 10^{14} \left( \frac{N}{10^{15}} \right) \left( \frac{M/M_0}{1000} \right)^{1/3} \left( \frac{R}{10^{10} \text{cm}} \right)^{1/2} \left( \frac{T}{10^4 K} \right)^{-5/2} \ll Re_{,\text{eff}}$$
Thus \( V_f \) is associated with macroscopic, instead of microscopic values.

Shakura & Sunyaev (1973)

Parameterized \( V_f = \alpha_{ss} C_s H \)

where \( H \) is dish height, \( C_s \) is sound speed and \( \alpha_{ss} \) is parameter.

\( \alpha_{ss} < 1 \) under assumption that, for disk which is vertically pressure supported, maximum random velocity is \( C_s \), (more on that later). Also, any structure must be \(< disk height \( H \). Thus \( \alpha_{ss} \leq 1 \), determining its exact value is an ongoing struggle.

Leading model is turbulence generated by magneto-rotational instability
(e.g. Balbus & Hawley, Rev Mod Phy 1998)

(Note also Blackman et al. 2006 for relation between \( \alpha \) and \( \beta = \frac{P_{rms}}{B^2/8\pi} \): robust \( \alpha = 0.2/\beta \) in many sims.)