\( X \) is microscopic quantity and \( \langle X \rangle \) is macroscopic. Thus, previous equation (where \( \langle X \rangle = \frac{1}{n} \int X f d^3 u \))

\[
\dot{\langle X \rangle} + \frac{\partial}{\partial x_i} \langle u_i \langle X \rangle \rangle - \langle u_i \frac{\partial X}{\partial x_i} \rangle - \frac{\dot{\langle F \rangle}}{m} + \frac{\partial}{\partial x_i} \langle F_i \rangle = 0
\]

(provides a link between microscopic and macroscopic quantities. Fluid equations are macro equations so (14b) is fundamental.

Recall that (14b) applies for any conserved quantity. Classically, mass is conserved, so lets first consider \( X = m \) in (14b) →
for $F$ independent of $u_i$ and all particles of same mass $m$:

$$\frac{∂}{∂t} (mn) + \frac{∂}{∂x_i} (nm \langle u_i \rangle) = 0$$  \hspace{1cm} (15)

if we write $p = mn$ and $V_i \equiv \langle u_i \rangle$

then we have continuity equation

$$\frac{∂p}{∂t} + \frac{∂}{∂x_i} (p V_i) = 0$$ \hspace{1cm} (16)

or

$$\frac{∂p}{∂t} + \nabla \cdot (p \mathbf{V}) = 0.$$  \hspace{1cm} (16)

This is one of the fundamental fluid equations.

Second Now let $x = mu_i$ in (14)

since $u_i, x_i$ are independent variables and $\frac{∂F}{∂u_i} = 0$ by assumption.

$$\frac{∂}{∂t} (nm \langle u_j \rangle) + \frac{∂}{∂x_i} (nm \langle u_i u_j \rangle) - n F_j = 0$$ \hspace{1cm} (17)

now define $p_{ij} = nm \langle (u_i - V_i)(u_j - V_j) \rangle$ with $V_i \equiv \langle u_i \rangle$

$$= nm \langle u_i u_j \rangle + nm V_i V_j - nm \langle u_j \rangle V_i + \overline{\langle u_j \rangle V_i}$$

$$= nm \langle u_i u_j \rangle - nm V_i V_j.$$  \hspace{1cm} (18)

thus (18) in (17) =>

$$\frac{∂}{∂t} (p \langle V_j \rangle) + \frac{∂}{∂x_i} p_{ij} + \frac{∂}{∂x_i} (p V_i V_j) - \frac{p F_j}{m} = 0$$ \hspace{1cm} (19)
Eqn (19) is the momentum equation with a pressure tensor.

The third let \( \lambda = \frac{1}{2} m | \mathbf{u} - \mathbf{v} |^2 \) in (14b).

This corresponds to conserved energy in collisions for monatomic gas, and constant mean velocity \( \mathbf{v} \).

The result is then:

\[
\frac{\partial \mathbf{v}}{\partial x_i} + \frac{\partial}{\partial x_i} (\mathbf{S} \mathbf{v}_i) + \frac{\partial \mathbf{v}_i}{\partial x_i} + \mathbf{P}_{ij} \Lambda_{ij} = 0 \tag{20}
\]

\( \mathbf{S} \mathbf{v}_i = \frac{1}{2} \langle | \mathbf{u} - \mathbf{v} |^2 \rangle \) is internal energy per mass

\( \mathbf{q} = \frac{\partial}{\partial x_i} (\mathbf{u} \cdot \mathbf{v}) \mathbf{v} \) = energy flux (units: energy/Arcsec/minute)

\( \Lambda_{ij} = \frac{1}{2} \left( \frac{\partial \mathbf{v}_i}{\partial x_j} + \frac{\partial \mathbf{v}_j}{\partial x_i} \right) \)

Now simplify (19) & (20) using (16)

The results are:

(19) \Rightarrow \quad \mathbf{S} \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}_i \frac{\partial \mathbf{v}}{\partial x_i} \right) = - \frac{\partial \mathbf{P}_{ij}}{\partial x_i} + \frac{\mathbf{q}}{m} \tag{21}

(20) \Rightarrow \quad \mathbf{S} \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}_i \frac{\partial \mathbf{v}}{\partial x_i} \right) + \frac{\partial \mathbf{v}_i}{\partial x_i} + \mathbf{P}_{ij} \Lambda_{ij} = 0 \tag{22}
Eqs (16), (21), (22) do represent mass, momentum, and energy conservation but these represent 5 eqns with 14 unknowns: \( \mathbf{V} \) (3 components)

\[ P_{ij} \] (6 components, since symmetric)

\[ f \] (1 component)

\[ q_i \] (3 components)

\[ E \] (1 component)

Thus we need relations between these quantities to close system of equations.

Eqs (16), (21), (22) are called the "moment" equations since they arise from multiplying a Boltzmann eqn by powers of \( O, i, j, k \) velocities and integrating over velocity.

Note distinction between \( u \) and \( \bar{v} \):

\[ \bar{v} \] mean velocity of overall flow

\[ v \] mean velocity of individual particle

\[ \bar{v} = \frac{1}{3} \sum_i v_i \]

Alternatively:

\[ \bar{v} \] overall flow

\[ v \] individual particle
We argued before, that collisions set up a Maxwellian distribution when frequent enough. Now let us see what this implies for reducing the number of variables, and a "simple" set of eqn:\n\[ (x,t) = n(x,t) \left[ \frac{m}{2 \pi k_b T(x,t)} \right]^{3/2} \exp \left[ - \frac{m(u - \dot{X}(x,t))^2}{2 k_b T(x,t)} \right] \]
where we write \( x,t \) dependencies explicitly.

Using (23) we have
\[ P_{ij} = \frac{n}{(2\pi k_b T)^{3/2}} \int d^3U \delta_{ij} \exp \left[ -\frac{mU^2}{2k_b T} \right] \]

Integral vanishes when integrand is odd \( \Rightarrow \)
\[ P_{ij} = \rho \delta_{ij} = \frac{n k_b T}{2} \]
which comes from integrating
\[ \int_0^\infty U^2 e^{-\frac{mU^2}{2k_b T}} \, dU = \frac{\sqrt{\pi} k_b T}{m} \]
We can also see that the flux $\dot{q}$ satisfies
\[ \dot{q} = 0, \] since it is odd integral.

From definition of $E = \frac{1}{2} \langle |V|^2 \rangle$

we also have that
\[ E = \frac{3}{2} \frac{\hbar T}{m} \]

Thus, using 24, 25, 26 we have
eliminated 3 variables of $\dot{q}$, 5 variables of the
original $Pij$ tensor, and $E$ can be written
as function of $p$, thus $14 - 9 = 5$ variables
left and 5 equations!

Using $Pij = p \delta_{ij}$ we also have
\[ Pij \Lambda_{ij} = \frac{1}{2} p \delta_{ij} \left( \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) = p \vec{\nabla} \cdot \vec{V} \]
from def of $\Lambda_{ij}$ below eqn (21).

Using (24), in (21) gives
\[ \frac{\partial V}{\partial t} + (\vec{V} - \vec{\nabla} E) \vec{V} = -\frac{1}{f} \int \vec{V} \, dp + \frac{E}{m} \]

Using (25), (26) & (27) in (22) gives
\[ \phi \left( \frac{\partial E}{\partial t} + \vec{V} \cdot \vec{\nabla} \phi \right) + p \nabla \cdot \vec{V} = 0 \]

resulting in
\[ \frac{\partial E}{\partial t} + \nabla \cdot (\phi \vec{V}) = 0 \]
Transport Processes:

1) In previous derivation $\frac{\partial}{\partial t} \rho = 0$ so no heat flow.
2) we also had $P_{ij}$ being diagonal.
   This means that momentum cannot be transported from one layer of fluid to another.
   This implies no shear forces.

Both 1 & 2 resulted from assumption of Maxwellian distribution; can immediately see that some departure from Maxwellian is required for transport:

\[
\text{Heat flows from hot to cold; in neighborhood of } P \text{ distribution is not isotropic and not Maxwellian!}
\]

we need to consider perturbations around Maxwellian distribution:

\[
f(x, u, t) = f^{(0)}(x, u, t) + g(x, u, t)
\]  

\[\uparrow \text{maxwellian} \uparrow \text{small departure}\]
Putting (31) in Boltzmann equation (page 6, of Jan 21 notes)

1. Collision integral is

\[ \int d^3u, d\Omega |\mathbf{u} - \mathbf{u}'| \sigma(\Omega) \left( f' f_1 - f f_1 \right) \]

\[ = \int d^3u, d\Omega |\mathbf{u} - \mathbf{u}'| \sigma(\Omega) \left( f^{(0)} g' + f^{(0)} g' - f^{(0)} g - f^{(0)} g \right) \]

to first order.

A typical term has magnitude

\[ - \int d^3u, d\Omega |\mathbf{u} - \mathbf{u}'| \sigma(\Omega) \left( f^{(0)} g \right) \leq -\mathbf{u}_{\text{rel}} n \sigma g(x, u, t) \]

\[ -\mathbf{u}_{\text{rel}} \text{no-l is a collision frequency with units} \]

\[ \frac{1}{c} \Rightarrow \text{collision integral is roughly} \]

\[ - \frac{\mathbf{g}}{c} \Rightarrow \text{Boltzmann eqn:} \]

\[ \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \frac{\mathbf{E}}{\mathbf{m}} \cdot \nabla \right) f = - \left( f - f^{(0)} \right) \]

(32)

This term is responsible for \( \frac{\partial}{\partial t} \),

when there are strong spatial gradients.

To order of mag

\[ |u| f^{(0)} \sim |g| \frac{1}{c} \]
where $L$ is gradient scale over which properties change.

\[ \frac{1}{f^{(0)}} = \frac{1}{\left| \mathbf{u} \right| T} \Rightarrow \frac{\lambda_{sp}}{L} = \alpha \]

\[ f = f^{(0)} + \alpha f^{(1)} + \alpha^2 f^{(2)} \]

Chapman-Enskog expansion.

To compute "corrections" use lowest order in (32).

\[ g = -2 \left( \frac{\partial f}{\partial \mathbf{u}} \right) ^{(0)} \frac{\partial \mathbf{u}}{\partial \mathbf{u}} + \frac{E_i}{m} \frac{\partial \mathbf{u}}{\partial \mathbf{u}} f^{(0)} \]

From (23), 
\[ f^{(0)} = \frac{m^{3/2} \mathbf{u}(x,t)}{(2\pi k_B T(x,t))^{3/2}} \exp \left( - \frac{m}{2k_B T(x,t)} \mathbf{u}^2 \right) \]

\[ f = f(n_j, T, \mathbf{v}) \]

So

\[ \frac{\partial f^{(0)}}{\partial t} + \frac{\partial f^{(0)}}{\partial \mathbf{u}} \mathbf{u} = 0 \]

\[ \frac{\partial f^{(0)}}{\partial \mathbf{v}} = \frac{2k_B T(x,t)}{m} \frac{\partial f^{(0)}}{\partial \mathbf{u}} \]

Using (23), for $f^{(0)}$ in (33), and using the $f^{(0)}$ "moment" equations for continuity (30), momentum (38), and energy density (39), we get. (set $F_i = 0$ to simplify)

\[ g = -2 \left( \frac{1}{2} \frac{2k_B T}{m} \mathbf{u} \left( \frac{m}{2k_B T} \mathbf{u}^2 - \frac{5}{2} \right) + \frac{m}{k_B T} \Lambda_{ij}(\mathbf{U}_i \mathbf{U}_j - \frac{1}{3} \mathbf{U} \cdot \mathbf{U}) \right) f^{(0)} \]

with $\Lambda_{ij} = \mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i$, $\mathbf{U} = \mathbf{u} - \mathbf{v}$
That \( g \) depends linearly on velocity and temperature gradients is expected based on our simple argument before, for deviations from Maxwellian dist. \( \to \) gradients imply deviation from Maxwellian.

\( \rightarrow \) Linear dependence on \( \tau \) implies that the longer the time between collisions, the more the deviation from Maxwellian can be sustained, and thus a larger correction \( g \). (collisions tend to make \( f \) closer to \( f^{(0)} \)).

\( \rightarrow \) Now we can calculate \( P_i, \tilde{g}, \) and \( \varepsilon \) for the non-Maxwellian distribution \( f = f^{(0)} + g \) with \( \langle A \rangle = \frac{1}{n} \int A f^{(0)} d^3u \) as defn for averaging of quantity \( A \), from before:

\[
\tilde{g} = \frac{n m}{2} \langle \hat{U} U^2 \rangle = \frac{g}{2} \int d^3U \hat{U} U^2 g
\]

Only even powers contribute to integrand so only 1st term on right of (34) contributes.

\( \gamma \hat{b} = -K \Delta T \), (where \( K = \frac{m^2 c T}{2 k_B T} \int d^3U U^4 \left( \frac{m}{2 k_B T} U^2 - \frac{5}{2} \right) f^{(0)} \))

\[
\gamma \hat{b} = \frac{5}{2} n \tau \frac{k_B^2 T}{m}
\]
That \( q = -K VT \) is a familiar form of heat transport equation (which we have derived from a "bottom up" approach).

Also:

\[ P_{ij} = \eta m \langle V_i V_j \rangle \]

is no longer diagonal instead:

\[ = \rho \delta_{ij} + \Pi_{ij} \]

with \( \Pi_{ij} = m \int d^3 V V_i V_j g \)

From (34) we then have

\[ \Pi_{ij} = -\frac{\Sigma m^2}{k_B T} \Lambda_{ki} \int d^3 V_i V_j (U_k U_e - \frac{1}{3} \delta_{ke} U^2) f^{(0)} \]

but for this integral, only isotropic contributions survive, since \( f^{(0)} \) is isotropic (no dependence on vector \( V \) only its magnitude).

this means \( \Rightarrow \)

\[ \langle V_i V_j U_k U_e \rangle = a \delta_{ij} \delta_{ke} + b \delta_{ij} \delta_{ke} + c \delta_{ie} \delta_{je} \]

\[ \langle V_i V_j \delta U_k^2 \rangle = d \delta_{ij} \delta_{ke} \]

\( \Rightarrow \)
to find $a, b, c$ need 3 equations.
Multiply by each separate $\delta$ combination:

\[
\langle U^4 \rangle = 9a + 3b + 3c \quad (35)
\]
\[
\langle U^4 \rangle = 3a + 9b + 3c \quad (36)
\]
\[
\langle U^4 \rangle = 3a + 3b + 9c \quad (37)
\]

\[
\Rightarrow \quad 0 = 6a - 6b
\]
\[
\Rightarrow \quad 0 = 6a - 6c
\]
\[
\Rightarrow \quad 0 = -6b - 6c
\]

\[
\Rightarrow \quad b = c = a = \frac{\langle U^4 \rangle}{15}
\]

also \( \langle U_i U_j U^2 \delta_{i,j} \rangle = d \delta_{ij} \delta_{i,j} \)

\[
\Rightarrow \quad 3\langle U^4 \rangle = 9d \Rightarrow \quad d = \frac{\langle U^4 \rangle}{3}
\]

\[
\Rightarrow \quad \sum_{\text{all}} \langle U_i U_j U_n U_e - U_i U_j U^2 \delta_{i,j} \rangle = \frac{\langle U^4 \rangle}{3}
\]

\[
= \sum_{\text{all}} \langle U^4 \rangle \left( \frac{\delta_{i,j} \delta_{n e}}{15} + \frac{\delta_{i, n} \delta_{i, e}}{15} + \frac{\delta_{i, e} \delta_{i, n}}{15} - \frac{1}{9} \delta_{i,j} \delta_{n e} \right)
\]

\[
= \frac{2}{15} \langle U^4 \rangle \sum_{\text{all}} \delta_{i,j} - \frac{6}{135} \sum_{\text{all}} \delta_{i,j} = \frac{2}{15} \langle U^4 \rangle (A_{ij} - \frac{1}{3} \delta_{i,j} \delta_{n e})
\]
Thus
\[ \Pi_{ij} \propto (\Lambda_{ij} - \frac{1}{3} \delta_{ij} \Lambda) \]
we can write
\[ \Pi_{ij} = -2M \left( \Lambda_{ij} - \frac{1}{3} \delta_{ij} \mathbf{D} \cdot \mathbf{V} \right) \]
\[ = \frac{1}{3} \Lambda \delta_{ij} \]

(38)

to get \( M \) evaluate one component of \( \Pi_{ij} \): (from p 35)
\[ \Pi_{11} = \frac{\tau m^2}{k_B T} \Lambda \text{me} \int d^3 \mathbf{U} U_1 U_2 \left( U_1 U_2 - \frac{1}{3} \text{trace} (U^2) \right) f^{(0)} \]
\[ = -2 \frac{\tau m^2}{k_B T} \Lambda \text{me} \int d^3 \mathbf{U} U_1^2 U_2^2 f^{(0)} \]
\[ \text{since only even powers to contribute.} \]
\[ \sqrt{2} \]

thus:
\[ M = \frac{m^2 \tau}{k_B T} \int d^3 \mathbf{U} U_1^2 U_2^2 f^{(0)} = 2 \pi k_B T \]

(38 a)

from (38) since \( \left< U_1^2 U_2^2 \right> = \frac{12 \tau^2}{m^2} \), use:
\[ \int_0^\infty e^{-a x^2} dx = \frac{\sqrt{\pi}}{a^{1/2}} \]

The off diagonal component of \( \Pi_{ij} \) has coefficient \( M \), this is viscosity! density

means momentum transport is possible between different fluids moving at different velocities.

Thus has coefficient \( M \), this is viscosity! density!
with expressions for \( \Phi \) and \( P_{ij} \)

we put them into the moment equations:

using \( P_{ij} \) and \( \Lambda_{ij} \equiv \frac{1}{3} \left( \partial_j \nu_i + 2 \partial_i \nu_j \right) \):

\[
\Pi_{ij} = -2m \left( \Lambda_{ij} - \frac{1}{3} \partial_j (\nabla \cdot \nu) \right)
\]

\[
\frac{\partial P_{ij}}{\partial x_j} = \frac{\partial P}{\partial x_i} - M \left[ \nabla^2 \nu_j + \frac{1}{3} \frac{\partial}{\partial x_j} (\nabla \cdot \nu) \right]
\]

then plugging into (19)

\[
\int \left( \frac{\partial \nu}{\partial t} + \nu \cdot \frac{\partial \nu}{\partial x} \right) = - \frac{\partial P}{\partial x_i} + M \left[ \nabla^2 \nu_j + \frac{1}{3} \frac{\partial}{\partial x_j} (\nabla \cdot \nu) \right] + \frac{\rho}{m} F_j
\]

from (38), (35a) & defn of \( \Lambda_{ij} \), we also have

\[
P_{ij} \Lambda_{ij} = p \nabla \cdot \nu - 2m \left[ \Lambda_{ij} \Lambda_{ij} - \frac{1}{3} (\nabla \cdot \nu)^2 \right]
\]

plugging (39) and (38) for \( \Pi_{ij} \) into energy moment eqn (20)

\[
\int \left( \frac{\partial \nu}{\partial t} + \nu \cdot \frac{\partial \nu}{\partial x} \right) = \nabla \cdot (K \nabla T) + p \nabla \cdot \nu - 2m \left[ \Lambda_{ij} \Lambda_{ij} - \frac{1}{3} (\nabla \cdot \nu)^2 \right] = 0
\]

now, \( m \) term in (40) and \( (\nabla \cdot \nu) \) term in (39)

are often small, if we neglect them
\[ \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla p + \vec{F} + \left( \frac{\partial \vec{M}}{\partial t} \right) \cdot \nabla \vec{V} \]  
\[ \int \left( \frac{\partial E}{\partial t} + V \cdot \nabla \varepsilon \right) - \nabla \cdot (J \varepsilon \nabla V) + \rho \nabla \cdot \vec{V} = 0 \]  
and mass continuity:
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \]

are the fluid equations, and we have now used \( \vec{F} \) to represent force density.