Spherically Symmetric Accretion

An inner object of mass $M$ accreting spherically from large gas cloud

- Goal is to predict $M$ for a given $f(r_0 \gg R_*)$, $T(r_0 \gg R_*)$ and surface boundary conditions on $R_*$

- Take spherical coords, and ignore $z$ momentum, so assume only radial dependence $u$ variables

- Steady flow

$\Rightarrow$ Mass conservation

$$D_r (p v) = 0$$

$$\frac{1}{r^2} \frac{d}{dr} (r^2 p v_r) = 0 \quad \Rightarrow \quad \text{Integrate}$$

$$\Rightarrow \quad r^2 p v = \text{constant}$$
Note: \( 4\pi r^2 g V_r = \dot{M} \)

\( V_r < 0 \Rightarrow \) accretion \( \dot{M} = \) accretion rate

\( V_r > 0 \Rightarrow \) ejection, wind, \( \dot{M} = \) ejection rate

\( \Rightarrow \) Next consider momentum equation

\( \text{Center equation}. \) Take \( \ddot{r} = \) gravitation force.

\[ F = -\frac{GMg}{r^2}, \quad \text{so momentum equation becomes:} \]

\[ V_r \frac{dV_r}{dr} + \frac{1}{\gamma} \frac{dp}{dr} + \frac{GM}{r^2} = 0 \]

\( \Rightarrow \) consider now the polytropic relation

\[ p = K \rho^\gamma \quad (1 < \gamma < \frac{5}{3}) \]

\( \uparrow \) isothermal \( \uparrow \) adiabatic for atmosphere gas

\( \text{(This replaces the energy equation.)} \)
for a real accretor, neither \( \gamma = 1 \) nor \( \gamma = 5/3 \) are exactly satisfied. The value \( \gamma \approx 5/3 \) is approached if the heating or cooling times are much larger than the infall time at that radius.

If we find \( P(r) \), \( g(r) \), then we get \( V(r) \) and we can use ideal gas relation \( T(r) = \frac{m_H M \rho}{3 \kappa_b} \).

Get temperature

\[
M = \frac{\sqrt{n_{mp}} + \sqrt{n_{ne}}}{n_{mp}} = \frac{1}{2}
\]

really need to integrate (124) but we can proceed quite far without integrating:

First write

\[
\frac{dP}{dr} = \frac{dP}{dg} \frac{dg}{dr} = C_s \frac{dP}{dr}
\]

Thus (124) becomes →
\[ V_r \frac{dV_r}{dr} + \frac{C_s^2}{r} \frac{ds}{dr} + \frac{GM}{r^2} = 0 \quad (126a) \]

but \( (123) \Rightarrow \)

\[ \frac{d \ln (r^2 V_r)}{dr} = 0 \]

\[ \Rightarrow \frac{1}{r^2} \frac{dV_r}{dr} = - \frac{1}{r^2 V_r} \frac{d}{dr} (r^2 V_r) \]

Using this in \( (125a) \Rightarrow \)

\[ V_r \frac{dV_r}{dr} - \frac{C_s^2}{r^2 V_r} \frac{d}{dr} (r^2 V_r) + \frac{GM}{r^2} = 0 \]

\[ \downarrow \]

\[ \frac{1}{2} \frac{dV_r^2}{dr} = - \frac{C_s^2}{V_r} \frac{dV_r}{dr} - \frac{2C_s^2}{r} + \frac{GM}{r} = 0 \]

\[ \frac{1}{2} \left( 1 - \frac{C_s^2}{V_r^2} \right) \frac{dV_r^2}{dr} = - \frac{GM}{r^2} \left[ 1 - \frac{2C_s^2 r}{6M} \right] \quad (127) \]

Useful equation
at large $r$, right side of (60) is positive because $c_s \to c_s(\infty)$ there.
On left side of (62) $c_s^2$,
\[
\frac{dV_r}{dr} < 0 \quad \text{for} \quad r \to \infty
\]
because gas is at rest there. ($V_r \to 0$)

$\Rightarrow \frac{c_s^2}{V_r} > 1$ \quad \text{for} \quad \text{large} \quad r$

that is, flow is subsonic at $r \to \infty$

for desired accretion solution.

As gas moves to smaller $r$,

the factor $(1 - \frac{2c_s^2}{GM})$ increases, provided

that $c_s^2$ increases more slowly than $\frac{1}{r}$

as $r$ is decreased. Note that

a factor vanishes at $r = \frac{GM}{2c_s^2(\infty)} = \frac{4GM}{c_s^2}$

the sonic radius \[\rightarrow\]
Thus provided $c_s^2$ increases slowly with $r$, the desired accretion solution will be supersonic $V_r > c_s^2$ at small $r$ (from left side of (127)). Note that approximating $c_s^2(r_{in}) = c_s^2(r_{out})$, we can use typical values for galaxy: $c_s^2(r_{in}) = \frac{kT_{in}}{\mu m_p} \approx 10^{12} \left(\frac{1}{10^4 \text{ K}}\right) \text{ cm}^2 \text{s}^{-2}$

then

\[ c_s = \sqrt{\frac{6M}{2c_s^2}} = \sqrt{\frac{6 \times 10^9}{2 \times 10^{14}}} = 3 \times 10^{20} \text{ cm} \]

\[ r_s = \frac{6M}{2c_s^2} = \frac{10^{-7} \times 6 \times 10^{39}}{2 \times 10^{14}} = 100 \text{ pc} \]

\[ 100 \text{ pc} \gg 10^6 \frac{GM}{c^2} = \left(\frac{10^{-7}}{10^{21}} \times 6 \times 10^{39}\right) = 6 \times 10^{11} \text{ cm} \]

Thus the sonic radius $\gg$ gravitational radius for B.H. in Galaxy

which helps justify assumption $c_s(r_{in}) = c_s(r_{in})$ since we can measure the temperatures at $r = 100 \text{ pc}$ and at $r = \text{ few kpc}$ and they don't differ hugely.
To be more rigorous, we really need to impose conditions at central object (star or BH) to fully specify soln.

Note again (13.7). at $r=R_s$

$V_r^2 = C_s^2$ or $\frac{dV_r}{dr} = 0$. Thus there are different possible solutions, depending so boundary conditions. There are 6 classes of solutions. The two most interesting are:

1. $V_r^2(R_s) = C_s^2(R_s)$, $V_r \to 0$ as $r \to \infty$
   - (a) $V_r^2 < C_s^2$, $r > R_s$
   - (b) $V_r^2 > C_s^2$, $r < R_s$

2. $V_r^2(R_s) = C_s^2(R_s)$, $V_r \to 0$ as $r \to 0$
   - (a) $V_r^2 > C_s^2$, $r > R_s$
   - (b) $V_r^2 < C_s^2$, $r < R_s$
Type I solution is called Bondi Accretion: flow goes from subsonic to supersonic as material falls inward.

Now let us go back and integrate (128)

$$\frac{V_r^2}{2} + \int \frac{dp}{p} - \frac{GM}{r} = \text{constant} \quad (128)$$

$$dp = K \gamma p^{\gamma-1} dp \quad \Rightarrow \quad \int \frac{dp}{p} = \int K \gamma p^{\gamma-2} dp \quad (128) \Rightarrow$$

$$\frac{V_r^2}{2} + K \gamma \frac{p^{\gamma-1}}{\gamma-1} - \frac{GM}{r} = \text{constant} \quad (129)$$

out $K \gamma p^{\gamma-1} = \frac{\gamma \rho}{p} = C_s^2$ (adiabatic sound speed)

$$\Rightarrow \quad (129) \Rightarrow$$

$$\frac{V_r^2}{2} + \frac{C_s^2}{\gamma-1} - \frac{GM}{r} = \text{const} \quad (130)$$
(62) is Bernoulli constant

Since type 1 soln $V_r \to 0$ as $r \to \infty$ (and $V_r = C_s$, at $r = r_s$)

we have from (130) that the constant

is $\frac{C_s^2(\infty)}{\gamma - 1}$ so at the sonic radius

$\left( \frac{C_s^2(r_s)}{\gamma - 1} \right)^{\frac{1}{2}} + \frac{1}{\gamma - 1} - 2 = \frac{C_s^2(\infty)}{\gamma - 1}$

$C_s(r_s) = C_s(\infty) \left( \frac{2}{5 - 3 \gamma} \right)^{\frac{1}{2}}$ (131)

(for $\gamma \approx 1.4$ or so, this

confirms $C_s(r_s) \ll C_s(\infty)$)

Now mass conservation gives

$8 \pi r^2 = \text{constant} \Rightarrow$

$M = 4 \pi r^2 p(-V_r) = 4 \pi r_s^2 p(r_s) C_s(r_s)$ (132)

and since

$C_s \propto \left( \frac{p}{\bar{p}} \right)^{\frac{1}{2}} (\gamma - 1)^{\frac{\gamma}{2}} \Rightarrow \frac{p(r_s)}{p(\infty)} = \left( \frac{C_s(r_s)}{C_s(\infty)} \right)^{\frac{2}{\gamma - 1}}$ (133)
Solving (131) and (133) in (132)

\[ \dot{M} = 4\pi r_s^2 g(\infty) \left[ \frac{2}{5-3\alpha} \right]^{1/2} C_5(\infty) \left( \frac{2}{5-3\alpha} \right)^{5/2} \]

\[ = 4\pi r_s^2 g(\infty) C_5(\infty) \left[ \frac{2}{5-3\alpha} \right]^{1/2} \frac{1+\alpha}{2(8-1)} \]

(134)

\[ r_s^2 \equiv \frac{G^2 M^2}{4 C_4(r_s)} \Rightarrow \frac{G^2 M^2}{4 C_4(\infty)} \left( \frac{5-3\alpha}{2} \right)^2 \]

Next compute \( V(r) \rightarrow \)
Using \( \dot{M} = 4\pi r^2 p V_r \)

we can find \( V_r(r) \) from \( C_s^2 \frac{\rho}{\gamma} \propto \rho r^{\gamma-1} \):

\[
V_r(r) = \frac{\dot{M}}{4\pi r^2 p(r)} = \frac{\dot{M}}{4\pi r^2 p(\infty)} \left[ \frac{C_s(\infty)}{C_s(r)} \right]^{\frac{\gamma}{\gamma-1}}
\]

Then substituting into Bernoulli’s integral

\[
\frac{V^2(r)}{2} + \frac{C_s^2(r)}{\gamma-1} - \frac{GM}{r} = \frac{C_s^2(\infty)}{\gamma-1}
\]  

(137)

gives \( C_s(r) \) and thus \( V(r) \).

The fractional exponents require numerical solution, but can infer behavior analytically.

- At large \( r \), quantities have asymptotic values \( V_r \to 0 \).
- At small \( r \), \( V(r) \) increases and supersedes \( C_s \) at \( r_s \), and then is balanced by \( \frac{GM}{r} \).

From (139)

\[
\frac{V^2(r)}{2} \leq \frac{C_s^2(r)}{\gamma-1}
\]  

(138)

\[
\Gamma \leq \frac{GM}{C_s^2(\infty)}
\]  

(137)
Let's analyze this a bit further:

1. \( V(r) \) approaches free-fall near the object then how does \( g(r) \) depend?

\[
4\pi G \rho r^2 V(r) = \dot{M} \\
\frac{1}{r^{1/2}} \text{ is free-fall} \\
\Rightarrow \dot{g} \propto r^{-3/2} \quad r < R_s
\]

\[
\Rightarrow \dot{T} \propto \frac{C_s^2}{g} \propto \frac{1}{r} \propto r^{-3} \quad r \gtrsim R_s
\]

(Note when radiation losses are considered at smaller \( r \), \( g \) is smaller and \( T \) may change more slowly, but let's stick with \( r = 1.4R_s \) here)

Since \( C_s^2 \propto r^{-0.6} \), when we plug into the left side of Bernoulli integral, the \( C_s^2 \) term increases less rapidly than the \( \frac{\dot{M}}{r} \) term as we move to smaller \( r \).

This justifies ignoring the \( C_s^2 \) term at small \( r \) and considering only balance between \( V^2 \) and \( \frac{\dot{M}}{r} \).
Thus our conclusion that
\[ \frac{V_r^2}{2} \geq GM \frac{1}{r} \]
justifies and this balance begins at \( r \leq r_{\text{acc}} = \frac{GM}{c_s^2(\infty)} \) (equation 138)

\( r_{\text{acc}} \) accretion radius.
The radius at which
thermal energy \( \geq \) grav. binding energy.

For \( r > r_{\text{acc}} \) gravity has little effect

For \( r < r_{\text{acc}} \) gravity dominates