Spherically Symmetric Accretion

- An object of mass $M$ accreting spherically from a large gas cloud
- Goal is to predict $M$ for a given $f(R_0 \gg R_*)$, $T(R_0 \gg R_*)$ and surface boundary conditions on $R_*$
- Take spherical co-ords, and ignore $\theta$ momentum, so assume only radial dependence w/ variables
- Steady flow

$\Rightarrow$ Mass conservation

$$\mathbf{D} \cdot (\rho \mathbf{v}) = 0$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho v_r \right) = 0 \Rightarrow \text{integrate}$$

$$\Rightarrow r^2 \rho v = \text{constant}$$
Note: \( 4 \pi r^2 g V_r = \dot{M} \)

\( V_r < 0 \Rightarrow \) accretion

\( V_r > 0 \Rightarrow \) ejection, wind

\( \dot{M} = \) accretion rate

\( \dot{M} = \) ejection rate

\( \Rightarrow \) Next consider momentum equation (Euler equation). Take \( F = \) gravitation force.

\[ F = -GM \frac{\dot{v}}{r^2} \]

so momentum equation becomes:

\[ V_r \frac{dV_r}{dr} + \frac{1}{2} \frac{dp}{dr} + \frac{GM}{r^2} = 0 \]

\[ (124) \]

Consider now the polytropic relation

\[ p = \rho \sigma \quad (1 < \gamma < \frac{5}{3}) \]

\[ \uparrow \quad \text{isothermal} \quad \text{adiabatic for nonatomic gas} \]

\( (\text{This replaces the energy equation} \ldots) \)
For a real accretor, neither \( \gamma = 1 \) nor \( \gamma = 5/3 \) are exactly satisfied. The value \( \gamma \approx 5/3 \) is approached if the heating or cooling times are much larger than the infall time at that radius.

If we find \( P(r) \), \( g(r) \) then we get \( V(r) \) and we can use ideal gas relation \( T(r) = \frac{m_A M_p}{3 k_B} \)

\[
\begin{align*}
M & = \frac{1}{2} n m_p + \frac{1}{2} \chi n m_e \\
& = \frac{1}{2} n m_p
\end{align*}
\]

We really need to integrate \((126)\) but we can proceed quite far without integrating:

First write

\[
\frac{dp}{dr} = \frac{dp}{dg} \frac{dg}{dr} = C_s \frac{dp}{dr}
\]

Thus \((124)\) becomes \( \cdots \)
\[ V_r \frac{dV_r}{dr} + \frac{C_s^2}{r} \frac{ds}{dr} + \frac{6M}{r^2} = 0 \quad (126a) \]

but \((123)\Rightarrow\)

\[ \frac{d}{dr} \ln (r^2 g V) = 0 \]

\[ \Rightarrow \frac{1}{s} \frac{ds}{dr} = -\frac{1}{r^2 Vr} \frac{d}{dr} (r^2 V) \]

Using this in \((126a)\Rightarrow\)

\[ V_r \frac{dV_r}{dr} - \frac{C_s^2}{r^2 Vr} \frac{d}{dr} (r^2 V) + \frac{6M}{r^2} = 0 \]

\[ \downarrow \]

\[ \frac{1}{2} \frac{dV_r^2}{dr} = \frac{C_s^2}{Vr} \frac{dV_r}{dr} - \frac{2C_s^2}{r} + \frac{6M}{r} = 0 \]

\[ \frac{1}{2} \left(1 - \frac{C_s^2}{Vr^2}\right) \frac{dV_r^2}{dr} = -\frac{6M}{r^2} \left[1 - \frac{2C_s^2}{6M} \right] \quad (127) \]

Useful equation
at large $r$, right side of (60) is positive because $c_s \to c_s(\infty)$ there. On left side of (64),
\[ \frac{dv_r}{dr} < 0 \quad \text{for} \quad r \to \infty \]
because gas is at rest there. ($v_r \to 0$)

$\Rightarrow \frac{c_s^2}{v_r^2} > 1 \quad \text{for large} \quad r$
that is, flow is subsonic at $r \to \infty$
for desired accretion solution.

As gas moves to smaller $r$,
the factor $(1-\frac{2c_s^2}{GM})$ increases, provided
that $c_s^2$ increases more slowly than $\frac{1}{r}$
as $r$ is decreased. Note that
$v$ is factor vanishes at $r = \frac{6M}{2c_s^2(\infty)} \equiv R_5,$
the sonic radius.
Thus provided $C_s^2$ increases slowly with $r$, the desired accretion soln will be supersonic $V_r > C_s^2$ at small $r$ (from left side of (127)).

Note that approximating $C_s^2(r_0) = C_s^2(r=g(s))$ we can use typical values for galaxy: $C_s^2(r_0) = \frac{4\pi G M}{M_p} \approx 10^{12} \left(\frac{1}{10^4}\right)^2 \frac{\text{cm}^2}{\text{s}^2}$

then

$$r_s = \frac{6M}{2C_s^2} = \frac{(10^{-7})(6 \times 10^{39})}{2 \times 10^{14}} \approx 3 \times 10^{20} \text{ cm}$$

$$100 \text{ pc} \gg r_s = \frac{6M}{C_s^2} = \left(\frac{10^{-7}}{10^{21}} \times 6 \times 10^{39}\right) \approx 6 \times 10^{11} \text{ cm}$$

Thus the sonic radius $\gg$ gravitational radius for B.H. in Galaxy, which helps justify assumption $C_s^2(r) = C_s^2(r_0)$ since we can measure the temperatures at $r=100 \text{ pc}$ and at $r=\text{few kpc}$ and they don't differ hugely.
To be more rigorous, we really need to impose conditions at a central object (star or BH) to fully specify the solution.

Note again (147), at $r = r_s$:

$v_r^2 = c_s^2$ or $\frac{d v_r}{dr} = 0$. Thus there are different possible solutions, depending on boundary conditions. There are 6 classes of solutions. The two most interesting are:

1. $v_r^2(r_s) = c_s^2(r_s)$, $v_r \to 0$ as $r \to \infty$
   - $v_r^2 < c_s^2$, $r > r_s$
   - $v_r^2 > c_s^2$, $r < r_s$

2. $v_r^2(r_s) = c_s^2(r_s)$, $v_r \to 0$ as $r \to 0$
   - $v_r^2 > c_s^2$, $r > r_s$
   - $v_r^2 < c_s^2$, $r < r_s$
Type 1 solution is called Bondi Accretion: flow goes from subsonic to supersonic as material falls inward.

Now let us go back and integrate (129)

\[ \frac{V_r^2}{2} + \int \frac{dp}{p} - \frac{GM}{r} = \text{constant} \]  (128)

\[ dp = K r^\gamma r^{-1} dr \Rightarrow \int \frac{dp}{p} = \int K r^\gamma r^{-2} dr \]

\[ (128) \Rightarrow \]

\[ \frac{V_r^2}{2} + K r^\gamma r^{-1} - \frac{GM}{r} = \text{constant} \]  (129)

\[ \Rightarrow K r^\gamma r^{-1} = \frac{\gamma P}{\rho} = C_s^2 \quad \text{adiabatic sound speed} \]

\[ (129) \Rightarrow \]

\[ \frac{V_r^2}{2} + \frac{C_s^2}{\gamma - 1} - \frac{GM}{r} = \text{const} \]  (130)
(6.2) is Bernoulli constant

Since type 1 soln \( V_r \to 0 \) as \( r \to \infty \) (and \( V_r = C_s \) at \( r = r_s \)) we have from (30) that the constant is \( \frac{C_s^2(\infty)}{\gamma - 1} \) so at the sonic radius

\[
\left[ \frac{1}{2} + \frac{1}{\gamma - 1} - 2 \right] = \frac{C_s^2(\infty)}{\gamma - 1}
\]

\[
C_s(r_s) = C_s(\infty) \left( \frac{2}{5 - 3 \gamma} \right)^{\frac{1}{\gamma - 1}} \tag{131}
\]

(for \( \gamma = 1.4 \) or so, this confirms \( C_s(r_s) < C_s(\infty) \))

Now mass conservation gives

\[
8 \pi r^2 = \text{constant} \implies \rho = \frac{M}{4 \pi r^2} = 4 \pi r_s^2 \rho(r_s) C_s(r_s) \tag{132}
\]

and since

\[
C_s \propto \left( \frac{p}{\rho} \right)^{\frac{1}{2}} \implies \rho \frac{\gamma - 1}{2} \Rightarrow \frac{\rho(r_s)}{\rho(\infty)} = \left[ \frac{C_s(r_s)}{C_s(\infty)} \right]^{\frac{2}{\gamma - 1}} \tag{133}
\]
Inserting (131) and (133) into (132)

\[ M = 4\pi r_s^2 g(\infty) \left[ \frac{2}{5-3\gamma} \right]^{1/2} C_s(\infty) \left( \frac{2}{5-3\gamma} \right)^{1/2} \]

\[ = 4\pi r_s^2 g(\infty) C_s(\infty) \left[ \frac{2}{5-3\gamma} \right]^{1/2} \frac{1+\gamma}{2(\gamma-1)} \]

\[ \text{out } r_s^2 = \frac{G^2 M^2}{4 C_s^4(r_s)} \Rightarrow \frac{G^2 M^2}{4 C_s^4(\infty)} \left( \frac{5-3\gamma}{2} \right)^2 \]

\[ p(\gamma) = \Rightarrow \]

\[ M = \pi G^{-2} M^2 g(\infty) \left( \frac{2}{5-3\gamma} \right) \]

\[ \left( \frac{1+\gamma}{2(\gamma-1)} \right) \frac{5-3\gamma}{2(\gamma-1)} \]

\[ \text{Next compute } \]

\[ V(r) \rightarrow \]

\[ (134) \]

\[ (135) \]
Using \( \dot{M} = 4 \pi r^2 \rho \, V_r \) previously derived, we can find \( V_r(r) \) from
\[
V_r(r) = \frac{\dot{M}}{4 \pi r^2 \rho (r)} = \frac{\dot{M}}{4 \pi r^2 \rho (\infty)} \left[ \frac{C_s(\infty)}{C_s(r)} \right]^{2/\gamma - 1}
\] (136)

Then substituting into Bernoulli integral
\[
\frac{V^2(r)}{2} + \frac{C_s^2(r)}{\gamma - 1} - \frac{GM}{r} = \frac{C_s^2(\infty)}{\gamma - 1}
\] (Bernoulli Integral) (137)
gives \( C_s(r) \) and thus \( V(r) \).

The fractional exponents require numerical solution, but can infer behavior analytically.

- At large \( r \), quantities have asymptote values \( \nu_r \to 0 \).
- At small \( r \), \( V(r) \) increases and supposes \( C_s \) at \( r_s \), and then is balanced by \( \frac{GM}{r} \).

From \( (139) \),
\[
\frac{V^2(r)}{2} \geq \frac{C_s^2(r)}{\gamma - 1}
\]
when approximately
\[
\Gamma \leq \frac{GM}{C_s^2(\infty)}
\] (138)
Let's analyze this a bit further:

1. \( V_r \) approaches free-fall near the object then how does \( g(r) \) depend?

### \( \frac{4\pi r^2 g}{V_r} = M \)

\[ \frac{1}{r^{12}} \text{ is free-fall} \]

\[ \Rightarrow g \propto r^{-3/2} \quad r < r_s \]

\[ \Rightarrow T \propto c_s^2 \propto \frac{1}{2} \int g^{3/2} = \frac{3}{2} \left( \frac{\sigma}{g} \right) \quad r^{-3} \quad r < r_s \]

(Note when radiation losses are considered at smaller \( r \), \( \sigma \) is smaller and \( T \) may change more slowly, but let's stick with \( \sigma = 1.4 \) here.)

Since \( c_s^2 \propto r^{-0.6} \), when we plug into left side of Bernoulli integral, the \( c_s^2 \) term increases less rapidly than the \( \frac{6M}{r} \) term as we move to smaller \( r \).

This justifies ignoring the \( c_s^2 \) term at small \( r \) and considering only balance between \( V^2 \) and \( \frac{6M}{r} \).
Hence our conclusion that

\[ \frac{V_r^2}{2} = G\frac{M}{r} \] at small \( r \) is justified and this balance begins at \( r \leq R_{\text{ac}} = \frac{G M}{c_s^2(\infty)} \) (equation 138)

\( R_{\text{ac}} \) denotes the accretion radius.

The radius at which thermal energy = grav. binding energy.

For \( r \gg R_{\text{ac}} \), gravity has little effect.

For \( r \ll R_{\text{ac}} \), gravity dominates.