Three-Dimensional Anisotropic Thermal Conduction Solver in AstroBEAR 3.0

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1 Anisotropic Thermal Conduction Equation

1.1 Equations

Thermal conduction in plasmas with the mean free path much larger than the gyroradius (also know as cyclotron radius, is the radius of the circular motion of a charged particle in the presence of a uniform magnetic field) is anisotropic with respect to the magnetic field line; heat flow primarily along the filed lines with little conduction in the perpendicular direction. In such cases, a divergence of anisotropic heat flux is added to the energy equation. Thermal conduction can modify the characteristic structure of the magnetohydrodynamic equations making it difficult to incorporate into upwind methods. However, thermal conduction can be evolved independently of the MHD equations using operator splitting, as done in [10]. The equation for the evolution of internal energy density due to anisotropic thermal conduction is

$$\vec{q} = \hat{b}n(\chi_{\parallel} - \chi_{\perp})(\hat{b}\cdot\nabla)T + n\chi_{\perp}\nabla T \quad (1)$$

where $\hat{b}$ is the unit vector along the field direction, n is the number density of electrons/ions?, χs are the temperature dependent thermal conductivities. The detailed expressions are

$$\chi_{\parallel} = \kappa_{\parallel} T^{5/2} \quad (2)$$

$$\chi_{\perp} = \kappa_{\perp}\frac{n}{B^2 T^{1/2}} \quad (3)$$

The heat diffusion equation in AstroBEAR

$$\frac{1}{\gamma-1}\rho C_v \frac{K_B\partial T}{\partial t} = \nabla\cdot\frac{1}{\gamma-1}\hat{q} \quad (4)$$

For simplicity we write

$$\frac{\partial T}{\partial t} = \nabla\cdot\hat{q} \quad (5)$$

1.2 Simpler Case Without Perpendicular Conductivity

We first consider the case:

$$\begin{cases} \chi_{\parallel} = \kappa_{\parallel} T^{5/2} \\ \chi_{\perp} = 0 \end{cases} \tag{6}$$

Here the parallel conductivity is much larger than the perpendicular conductivity (larger by a factor of 10^9). These expressions are from the Orlando simulation. In a different scenario (i.e. when considering some dense and high pressure plasma), the expressions may be different. The heat flux is

$$\vec{q} = \hat{b} n \chi_{\parallel} (\hat{b} \cdot \nabla T) \tag{7}$$

So

$$\begin{aligned} \nabla \cdot \vec{q} &= \nabla \cdot \left[\hat{b} n \chi_{\parallel} (\hat{b} \cdot \nabla T) \right] \\ &= n \chi_{\parallel} (\hat{b} \cdot \nabla)^2 T + n (\hat{b} \cdot \nabla T)(\hat{b} \cdot \nabla \chi_{\parallel}) \end{aligned} \tag{8}$$

The first part in 8 is

$$\begin{aligned} n \chi_{\parallel} (\hat{b} \cdot \nabla)^2 T = & n \chi_{\parallel} \left[\left(b_x^2 \frac{\partial^2 T}{\partial x^2} + b_x b_y \frac{\partial^2 T}{\partial x \partial y} + b_x b_z \frac{\partial^2 T}{\partial x \partial z} \right) + \right. \\ & \left(b_y b_x \frac{\partial^2 T}{\partial x \partial y} + b_y^2 \frac{\partial^2 T}{\partial y^2} + b_y b_z \frac{\partial^2 T}{\partial y \partial z} \right) + \\ & \left. \left(b_z b_x \frac{\partial^2 T}{\partial z \partial x} + b_z b_y \frac{\partial^2 T}{\partial z \partial y} + b_z^2 \frac{\partial^2 T}{\partial z^2} \right) \right] \end{aligned} \tag{9}$$

And since from Eq. 2

$$\hat{b} \cdot \nabla \chi_{\parallel} = 2.5 \kappa_{\parallel} T^{3/2} (\hat{b} \cdot \nabla T) \tag{10}$$

So the second part in Eq. 8 is

$$\begin{aligned} & n(\hat{b} \cdot \nabla T)(\hat{b} \cdot \nabla \chi_{\parallel}) \\ & = 2.5 n \kappa_{\parallel} T^{3/2} (\hat{b} \cdot \nabla T)^2 \\ & = 2.5 n \kappa_{\parallel} T^{3/2} \left[b_x^2 (\frac{\partial T}{\partial x})^2 + b_y^2 (\frac{\partial T}{\partial y})^2 + b_z^2 (\frac{\partial T}{\partial z})^2 \right. \\ & \left. + \; 2 b_x b_y \frac{\partial T}{\partial x} \frac{\partial T}{\partial y} + 2 b_x b_z \frac{\partial T}{\partial x} \frac{\partial T}{\partial z} + 2 b_y bzy \frac{\partial T}{\partial y} \frac{\partial T}{\partial z} \right] \end{aligned} \tag{11}$$

So Eq. 5 becomes

$$
\begin{aligned}
\frac{\partial T}{\partial t} =& 2.5\kappa_{\parallel} T^{3/2} n \left[b_x^2 \left(\frac{\partial T}{\partial x}\right)^2 + b_y^2 \left(\frac{\partial T}{\partial y}\right)^2 + b_z^2 \left(\frac{\partial T}{\partial z}\right)^2 + \right. \\
& \left. 2b_x b_y \frac{\partial T}{\partial x}\frac{\partial T}{\partial y} + 2b_x b_z \frac{\partial T}{\partial x}\frac{\partial T}{\partial z} + 2b_y b_z \frac{\partial T}{\partial y}\frac{\partial T}{\partial z} \right] + \\
& n\kappa_{\parallel} T^{2.5} \left[\left(b_x^2 \frac{\partial^2 T}{\partial x^2} + 2b_x b_y \frac{\partial^2 T}{\partial x \partial y} + 2b_x b_z \frac{\partial^2 T}{\partial x \partial z} \right) + \right. \\
& \left(b_y^2 \frac{\partial^2 T}{\partial y^2} + 2b_y b_z \frac{\partial^2 T}{\partial y \partial z} \right) + \\
& \left. \left(b_z^2 \frac{\partial^2 T}{\partial z^2} \right) \right]
\end{aligned} \tag{12}
$$

1.3 With Perpendicular Conductivity

Then we consider the case:

$$\chi_{\parallel} = \kappa_{\parallel} T^{5/2} \tag{13}$$

$$\chi_{\perp} = \kappa_{\perp} \frac{n}{B^2 T^{1/2}} \tag{14}$$

From Eq. 1 we have

$$\nabla \cdot \vec{q} = \nabla \cdot \left[\hat{b} n (\chi_{\parallel} - \chi_{\perp}) \left(\hat{b} \cdot \nabla T \right) \right] + \chi_{\perp} n \nabla^2 T \tag{15}$$

Since $\chi_{\parallel}$ is a function of T and $\chi_{\perp}$ is a function of B, use Einstein's index we have

$$
\begin{aligned}
\partial_i q_i =& n(\chi_{\parallel} - \chi_{\perp})(\partial_i b_i)(b_j \partial_j T) \\
& + n[\partial_T(\chi_{\parallel} - \chi_{\perp})(b_i \partial_i T) - (\partial_B \chi_{\perp})\partial_i B](b_j \partial_j T_j) \\
& + n(\chi_{\parallel} - \chi_{\perp})[b_i(\partial_i b_j)\partial_j T + b_i b_j \partial_i \partial_j T] \\
& + \chi_{\perp} n \partial_i \partial_i T_i
\end{aligned} \tag{16}
$$

Regroup Eq.16 according to the derivatives of temperature T, we have

$$
\begin{aligned}
\partial_i q_i =& [n(\chi_{\parallel} - \chi_{\perp})(\partial_i b_i) - n b_i b_j \partial_B \chi_{\perp} \partial_j B](b_j \partial_j T) \\
& + n\partial_T(\chi_{\parallel} - \chi_{\perp})(b_i \partial_i T)(b_j \partial_j T) \\
& + n(\chi_{\parallel} - \chi_{\perp}) b_i (\partial_i b_j) \partial_j T \\
& + n(\chi_{\parallel} - \chi_{\perp}) b_i b_j (1 - \delta_{i,j}) \partial_i \partial_j T \\
& + 3n(\chi_{\parallel} - \chi_{\perp}) b_i^2 \partial_i^2 T \\
& + \chi_{\perp} n \partial_i \partial_i T
\end{aligned} \tag{17}
$$

And we define the following coefficients

$$
\begin{aligned}
A_i &= n(\chi_\parallel - \chi_\perp)(\partial_j b_j) b_i - n b_i b_j \partial_B \chi_\perp \partial_j B \\
B_{i,j} &= n b_i b_j \partial_T (\chi_\parallel - \chi_\perp) \\
C_i &= b_i n(\chi_\parallel - \chi_\perp)(\partial_j b_i) \\
D_{i,j} &= b_i b_j n(\chi_\parallel - \chi_\perp)(1 - \delta_{i,j}) \\
E_i &= 3 b_i^2 n(\chi_\parallel - \chi_\perp) \\
F &= \chi_\perp n
\end{aligned}
\tag{18}
$$

So Eq.17 becomes

$$
\partial_i q_i = A_i(\partial_i T) + B_{ij}(\partial_i T)(\partial_j T) + C_i \partial_i T + D_{ij} \partial_i \partial_j T + (E_i + F)\partial_i^2 T \tag{19}
$$

1.4 Discretization and Linearization

$$
\begin{aligned}
T^{n+1} - T^n = (\frac{\Delta t}{4\Delta x^2})&(A_i + C_i)(T_{+i} - T_{-i}) \\
&+ B_{i,j}(T_{+i} - T_{-i})(T_{+j} - T_{-j}) \\
&+ D_{i,j}(T_{+i,+j} - T_{+i,-j} - T_{-i,+j} + T_{-i,-j}) \\
&+ (E_i + F_j)(T_{+i} - 2T + T_{-i})
\end{aligned}
\tag{20}
$$

$$
B_{i,j}[(T^*_{+j} - T^*_{-j})(T_{+i} - T_{-i}) + (T^*_{+i} - T^*_{-i})(T_{+j} - T_{-j}) - (T^*_{+i} - T^*_{-i})(T^*_{+j} - T^*_{-j}) \tag{21}
$$

$$
\begin{aligned}
&T_{+i}[A_i + C_i + E_i + F_i + 2B_{i,j}(T^*_{+j} - T^*_{-j})] \\
&T_{-i}[-A_i - C_i + E_i + F_i - 2B_{i,j}(T^*_{+j} - T^*_{-j})] \\
&T_0[-6(E_i + F_i) - \frac{\Delta t}{4\Delta x^2} \\
&T_{\pm i,\pm j}[D_{i,j}] \\
&RHS[B_{i,j}(T^*_{+i} - T^*_{-i})(T^*_{+j} - T^*_{-j})]
\end{aligned}
\tag{22}
$$

1.5 Derivative of Magnetic Field is Neglectible

Since

$$
\hat{b} \cdot \nabla(\chi_\parallel - \chi_\perp) = \left[2.5\kappa_\parallel T^{3/2} + 0.5\kappa_\perp \frac{n}{B^2 T^{3/2}}\right](\hat{b} \cdot \nabla T) \tag{23}
$$

so similar as 12 the first term on the right of Eq.15 is

$$
\begin{aligned}
&n(\hat{b}\cdot\nabla T)\left[\hat{b}\cdot\nabla(\chi_{\parallel}-\chi_{\perp})\right]\\
&=n\left(2.5\kappa_{\parallel}T^{3/2}+0.5\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)(\hat{b}\cdot\nabla T)^2\\
&=n\left(2.5\kappa_{\parallel}T^{3/2}+0.5\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)\left[b_x^2\left(\frac{\partial T}{\partial x}\right)^2+b_y^2\left(\frac{\partial T}{\partial y}\right)^2+b_z^2\left(\frac{\partial T}{\partial z}\right)^2+\right.\\
&\left.2b_xb_y\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}+2b_xb_z\frac{\partial T}{\partial x}\frac{\partial T}{\partial z}+2b_yb_z\frac{\partial T}{\partial y}\frac{\partial T}{\partial z}\right]
\end{aligned}
\tag{24}
$$

and the second term on the right of Eq.15 is

$$
\begin{aligned}
&n(\chi_{\parallel}-\chi_{\perp})(\hat{b}\cdot\nabla)^2T\\
&=n\left(\kappa_{\parallel}T^{5/2}-\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right)(\hat{b}\cdot\nabla)^2T\\
&=n\left(\kappa_{\parallel}T^{5/2}-\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right)\left[\left(b_x^2\frac{\partial^2T}{\partial x^2}+b_xb_y\frac{\partial^2T}{\partial x\partial y}+b_xb_z\frac{\partial^2T}{\partial x\partial z}\right)+\right.\\
&\left(b_yb_x\frac{\partial^2T}{\partial y\partial x}+b_y^2\frac{\partial^2T}{\partial y^2}+b_yb_z\frac{\partial^2T}{\partial y\partial z}\right)+\\
&\left.\left(b_zb_x\frac{\partial^2T}{\partial z\partial x}+b_zb_y\frac{\partial^2T}{\partial z\partial y}+b_z^2\frac{\partial^2T}{\partial z^2}\right)\right]
\end{aligned}
\tag{25}
$$

and finally the third term on the right of Eq.15 is

$$
n\chi_{\perp}\nabla^2T=n\kappa_{\perp}\frac{n}{B^2T^{1/2}}\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}+\frac{\partial^2T}{\partial z^2}\right)
\tag{26}
$$

So Eq. 5 becomes

$$
\begin{aligned}
\frac{\partial T}{\partial t}=&n\left(2.5\kappa_{\parallel}T^{3/2}+0.5\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)\left[b_x^2\left(\frac{\partial T}{\partial x}\right)^2+b_y^2\left(\frac{\partial T}{\partial y}\right)^2+b_z^2\left(\frac{\partial T}{\partial z}\right)^2+\right.\\
&\left.2b_xb_y\frac{\partial T}{\partial x}\frac{\partial T}{\partial y}+2b_xb_z\frac{\partial T}{\partial x}\frac{\partial T}{\partial z}+2b_yb_z\frac{\partial T}{\partial y}\frac{\partial T}{\partial z}\right]+\\
&n\left(\kappa_{\parallel}T^{5/2}-\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right)\left[\left(b_x^2\frac{\partial^2T}{\partial x^2}+2b_xb_y\frac{\partial^2T}{\partial x\partial y}+2b_xb_z\frac{\partial^2T}{\partial x\partial z}\right)+\right.\\
&+\left(b_y^2\frac{\partial^2T}{\partial y^2}+2b_yb_z\frac{\partial^2T}{\partial y\partial z}\right)+\\
&\left.\left(b_z^2\frac{\partial^2T}{\partial z^2}\right)\right]+\\
&n\kappa_{\perp}\frac{n}{B^2T^{1/2}}\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}+\frac{\partial^2T}{\partial z^2}\right)
\end{aligned}
\tag{27}
$$

The following Table gives the coefficients for each derivative term

$$
\begin{aligned}
&\frac{\partial^2 T}{\partial x^2} && n\left[b_x^2\kappa_{\parallel}T^{5/2}+(1-b_x^2)\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right]=A_{11}\\
&\frac{\partial^2 T}{\partial y^2} && n\left[b_y^2\kappa_{\parallel}T^{5/2}+(1-b_y^2)\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right]=A_{22}\\
&\frac{\partial^2 T}{\partial z^2} && n\left[b_z^2\kappa_{\parallel}T^{5/2}+(1-b_z^2)\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right]=A_{33}\\
&\frac{\partial^2 T}{\partial x\partial y} && 2nb_xb_y\left(\kappa_{\parallel}T^{5/2}-\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right)=A_{12}\\
&\frac{\partial^2 T}{\partial x\partial z} && 2nb_xb_z\left(\kappa_{\parallel}T^{5/2}-\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right)=A_{13}\\
&\frac{\partial^2 T}{\partial y\partial z} && 2nb_yb_z\left(\kappa_{\parallel}T^{5/2}-\kappa_{\perp}\frac{n}{B^2T^{1/2}}\right)=A_{23}\\
&\left(\frac{\partial T}{\partial x}\right)^2 && nb_x^2\left(2.5\kappa_{\parallel}T^{3/2}+0.5\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)=B_{11}\\
&\left(\frac{\partial T}{\partial y}\right)^2 && nb_y^2\left(2.5\kappa_{\parallel}T^{3/2}+0.5\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)=B_{22}\\
&\left(\frac{\partial T}{\partial z}\right)^2 && nb_z^2\left(2.5\kappa_{\parallel}T^{3/2}+0.5\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)=B_{33}\\
&\frac{\partial T}{\partial x}\frac{\partial T}{\partial y} && nb_xb_y\left(5\kappa_{\parallel}T^{3/2}+\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)=B_{12}\\
&\frac{\partial T}{\partial x}\frac{\partial T}{\partial z} && nb_xb_z\left(5\kappa_{\parallel}T^{3/2}+\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)=B_{13}\\
&\frac{\partial T}{\partial y}\frac{\partial T}{\partial z} && nb_yb_z\left(5\kappa_{\parallel}T^{3/2}+\kappa_{\perp}\frac{n}{B^2T^{3/2}}\right)=B_{23}
\end{aligned}
\tag{28}
$$

With the coefficients above, Eq. 27 can be re-written as

$$
\frac{\partial T}{\partial t}=\sum_{i,j=1,2,3}\left[A_{ij}\frac{\partial^2 T}{\partial x_i\partial x_j}+B_{i,j}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}\right] \tag{29}
$$

1.6 Discretization and Linearization

Assume $\Delta x=\Delta y=\Delta z=a$. Discretizing Eq.27 using the coefficients in Table. 28 we have

$$
\begin{aligned}
\frac{T^{l+1}_{i,j,k}-T^{l}_{i,j,k}}{\Delta t} =& A_{11}\frac{T^{l}_{i+1,j,k}+T^{l}_{i-1,j,k}-2T^{l}_{i,j,k}}{a^2}+\\
& A_{22}\frac{T^{l}_{i,j+1,k}+T^{l}_{i,j-1,k}-2T^{l}_{i,j,k}}{a^2}+\\
& A_{33}\frac{T^{l}_{i,j,k+1}+T^{l}_{i,j,k-1}-2T^{l}_{i,j,k}}{a^2}+\\
& A_{12}\frac{T^{l}_{i+1,j+1,k}-T^{l}_{i-1,j+1,k}-T^{l}_{i+1,j-1,k}+T_{i-1,j-1,k}}{2a^2}+\\
& A_{23}\frac{T^{l}_{i,j+1,k+1}-T^{l}_{i,j+1,k-1}-T^{l}_{i,j-1,k+1}+T_{i,j-1,k-1}}{2a^2}+\\
& A_{13}\frac{T^{l}_{i+1,j,k+1}-T^{l}_{i+1,j,k-1}-T^{l}_{i-1,j,k+1}+T_{i-1,j,k-1}}{2a^2}+\\
& B_{11}\frac{T^{l}_{i+1,j,k}T^{l}_{i+1,j,k}-2T^{l}_{i+1,j,k}T^{l}_{i-1,j,k}+T^{l}_{i-1,j,k}T^{l}_{i-1,j,k}}{4a^2}+\\
& B_{22}\frac{T^{l}_{i,j+1,k}T^{l}_{i,j+1,k}-2T^{l}_{i,j+1,k}T^{l}_{i,j-1,k}+T^{l}_{i,j-1,k}T^{l}_{i,j-1,k}}{4a^2}+\\
& B_{33}\frac{T^{l}_{i,j,k+1}T^{l}_{i,j,k+1}-2T^{l}_{i,j,k+1}T^{l}_{i,j,k-1}+T^{l}_{i,j,k-1}T^{l}_{i,j,k-1}}{4a^2}+\\
& B_{12}\frac{T^{l}_{i+1,j,k}T^{l}_{i,j+1,k}-T^{l}_{i+1,j,k}T^{l}_{i,j-1,k}-T^{l}_{i-1,j,k}T^{l}_{i,j+1,k}+T^{l}_{i-1,j,k}T^{l}_{i,j-1,k}}{4a^2}+\\
& B_{13}\frac{T^{l}_{i+1,j,k}T^{l}_{i,j,k+1}-T^{l}_{i+1,j,k}T^{l}_{i,j,k-1}-T^{l}_{i-1,j,k}T^{l}_{i,j,k+1}+T^{l}_{i-1,j,k}T^{l}_{i,j,k-1}}{4a^2}+\\
& B_{23}\frac{T^{l}_{i,j+1,k}T^{l}_{i,j,k+1}-T^{l}_{i,j+1,k}T^{l}_{i,j,k-1}-T^{l}_{i,j-1,k}T^{l}_{i,j,k+1}+T^{l}_{i,j-1,k}T^{l}_{i,j,k-1}}{4a^2}
\end{aligned}
\tag{30}
$$

Or in a different form

$$
\begin{aligned}
&\frac{T^{l+1}_{i,j,k}-T^{l}_{i,j,k}}{\Delta t}\\
&=\sum_{\mu=1,2,3} A_{\mu\mu}\frac{T^{l}_{i+\delta_{\mu,1},j+\delta_{\mu,2},k+\delta_{\mu,3}}+T^{l}_{i-\delta_{\mu,1},j-\delta_{\mu,2},k-\delta_{\mu,3}}-2T^{l}_{i,j,k}}{a^2}\\
&+\sum_{\mu,\nu=1,2,3}^{\mu<\nu}\frac{T^{l}_{i+\delta_{\mu,1},j+\delta_{\mu,2}+\delta_{\nu,2},k+\delta_{\nu,3}}-T^{l}_{i+\delta_{\mu,1},j+\delta_{\mu,2}-\delta_{\nu,2},k-\delta_{\nu,3}}-T^{l}_{i-\delta_{\mu,1},j-\delta_{\mu,2}+\delta_{\nu,2},k+\delta_{\nu,3}}+T^{l}_{i-\delta_{\mu,1},j-\delta_{\mu,2}-\delta_{\nu,2},k-\delta_{\nu,3}}}{a^2}\\
&+
\end{aligned}
\tag{31}
$$

To linearize the non-linear terms in Eq.30 we use Taylor expansion of the term at time $l+1$. For example, for the terms of B_{11} we have

$$\begin{aligned}\left(T^{l+1}_{i\pm1,j,k}\right)^2 &= (T^l_{i\pm1,j,k} + 2T^l_{i,j,k}(T^{l+1}_{i\pm1,j,k} - T^l_{i\pm1,j,k}) \\ &= 2T^l_{i\pm1,j,k}T^{l+1}_{i\pm1,j,k} - (T^l_{i\pm1,j,k})^2\end{aligned} \tag{32}$$

$$\begin{aligned}2T^{l+1}_{i+1,j,k}T^{l+1}_{i-1,j,k} &= 2\left[T^l_{i+1,j,k} + (T^{l+1}_{i+1,j,k} - T^l_{i+1,j,k})\right]\left[T^l_{i-1,j,k} + (T^{l+1}_{i-1,j,k} - T^l_{i-1,j,k})\right] \\ &\approx 2\left[T^l_{i+1,j,k}T^l_{i-1,j,k} + T^l_{i+1,j,k}(T^{l+1}_{i-1,j,k} - T^l_{i-1,j,k}) + (T^{l+1}_{i+1,j,k} - T^l_{i+1,j,k})T^l_{i-1,j,k}\right] \\ &= 2\left[T^l_{i+1,j,k}T^{l+1}_{i-1,j,k} + T^l_{i-1,j,k}T^{l+1}_{i+1,j,k} - T^l_{i+1,j,k}T^l_{i-1,j,k}\right]\end{aligned} \tag{33}$$

Where we dropped the higher order terms..

We do the same for the B_{22} and B_{33} terms:

$$\left(T^{l+1}_{i,j\pm1,k}\right)^2 = 2T^l_{i,j\pm1,k}T^{l+1}_{i,j\pm1,k} - (T^l_{i,j\pm1,k})^2 \tag{34}$$

$$2T^{l+1}_{i,j+1,k}T^{l+1}_{i,j-1,k} = 2\left[T^l_{i,j+1,k}T^{l+1}_{i,j-1,k} + T^l_{i,j-1,k}T^{l+1}_{i,j+1,k} - T^l_{i,j+1,k}T^l_{i,j-1,k}\right] \tag{35}$$

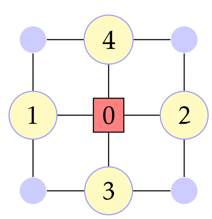
$$\left(T^{l+1}_{i,j,k\pm1}\right)^2 = 2T^l_{i,j,k\pm1}T^{l+1}_{i,j,k\pm1} - (T^l_{i,j,k\pm1})^2 \tag{36}$$

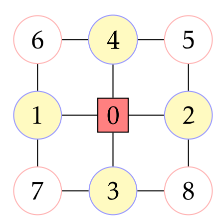
$$2T^{l+1}_{i,j,k+1}T^{l+1}_{i,j,k-1} = 2\left[T^l_{i,j,k+1}T^{l+1}_{i,j,k-1} + T^l_{i,j,k-1}T^{l+1}_{i,j,k+1} - T^l_{i,j,k+1}T^l_{i,j,k-1}\right] \tag{37}$$

2 Anisotropic Solver in AstroBEAR

2.1 Two Dimensional

In 2D, if it's isotropic conductivity, the neighbor index are shown on the left below and those for the anisotropic conductivity case are shown on the right.





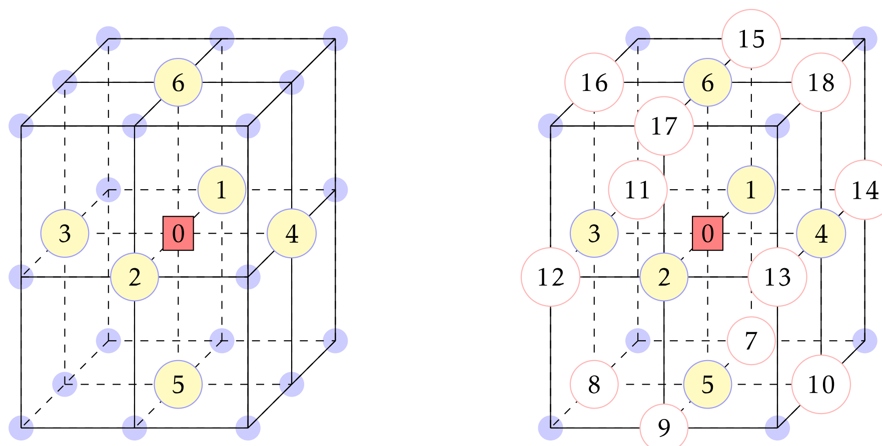
So corresponding the offsets matrix in the code for 2D case is

```
offsets(:,1)=(/-1,0/)
offsets(:,2)=(/1,0/)
offsets(:,3)=(/0,-1/)
```

```
offsets(:,4)=(/0,1/)
offsets(:,5)=(/1,1/)
offsets(:,6)=(/-1,1/)
offsets(:,7)=(/-1,-1/)
offsets(:,8)=(/1,-1/)
```

2.2 Three Dimensional

In 3D, if it's isotropic conductivity, the neighbor index are shown on the left below and those for the anisotropic conductivity case are shown on the right.



So corresponding the offsets matrix in the code for 3D case is

```
offsets(:,1)=(/-1,0,0/)
offsets(:,2)=(/1,0,0/)
offsets(:,3)=(/0,-1,0/)
offsets(:,4)=(/0,1,0/)
offsets(:,5)=(/0,0,-1/)
offsets(:,6)=(/0,0,1/)
offsets(:,7)=(/-1,0,-1/)
offsets(:,8)=(/0,-1,-1/)
offsets(:,9)=(/1,0,-1/)
offsets(:,10)=(/0,1,-1/)
offsets(:,11)=(/-1,-1,0/)
offsets(:,12)=(/1,-1,0/)
offsets(:,13)=(/1,1,0/)
offsets(:,14)=(/-1,1,0/)
offsets(:,15)=(/-1,0,1/)
offsets(:,16)=(/0,-1,1/)
offsets(:,17)=(/1,0,1/)
offsets(:,18)=(/0,1,1/)
```

In the code we define array neighbor(dir, edge, subdir, subedge) according the the diagram above. The subdir is defined using function $mod(l+q-1, nDim)+1$. So in 2D

$$subdir = \begin{cases} \text{2 or y, for x(l=1,q=1)} \\ \text{1 or x, for y(l=2,q=2)} \end{cases} \tag{38}$$

and in 3D, since every direction has two sub-directions (for example, x direction has y and z as sub-directions)

$$subdir = \begin{cases} \text{2 or y, for x (l=1, q=1)} \\ \text{3 or z, for x (l=1, q=2)} \\ \text{3 or z, for y (l=2, q=1)} \\ \text{1 or x, for y (l=2, q=2)} \\ \text{1 or x, for z (l=3, q=1)} \\ \text{2 or y, for z (l=3, q=2)} \end{cases} \tag{39}$$

3 Multiphysics Integration