Classical Mechanics - Module 1

Welcome to the PHY235 workshop! The purpose of the workshop is to review some of the more challenging aspects of the course, as well as explore some areas that you were unable to tackle in class due to time restrictions. Throughout the workshop, you will work on some problems individually, some problems in small groups, and occasionally you will tackle a problem as a large group. Studies have shown that learning is enhanced when students work together, so let the learning begin! Take a moment to introduce yourself and meet everyone else in your workshop section.

Do problems 1, 2, 4, and 5 at the workshop sections held during the week of 7 September 2015.

1. The TA will assign a problem (or two, depending on the number of people present) to each person and a space will be provided to write the answer on the board. Once you have completed your problem, check with a classmate before writing it on the board. After you have verified that you have found the correct solution, write your answer in the space provided on the board, taking care to include the steps that you used to arrive at your solution.

   You will need the following information:

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 9\mathbf{k} \qquad \mathbf{b} = -2\mathbf{i} + 3\mathbf{k}$$

$$\mathbf{c} = -2\mathbf{i} + \mathbf{j} - 6\mathbf{k} \qquad \mathbf{d} = \mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{E} = \begin{pmatrix} 2 & 7 & -4 \\ 3 & 1 & -2 \\ -2 & 0 & 5 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} 2 & -4 \\ 7 & 1 \\ -1 & 1 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} -8 & -1 & -3 \\ -4 & 2 & -2 \\ -1 & 0 & 0 \end{pmatrix}$$

   Calculate each of the following:

   (a) $|\mathbf{a} - (\mathbf{b} + 3\mathbf{c})|$

   (b) component of $\mathbf{c}$ along $\mathbf{a}$

   (c) angle between $\mathbf{c}$ and $\mathbf{d}$

   (d) $(\mathbf{b} \times \mathbf{d}) \cdot \mathbf{a}$

   (e) $(\mathbf{b} \times \mathbf{d}) \times \mathbf{a}$

   (f) $\mathbf{b} \times (\mathbf{d} \times \mathbf{a})$

   (g) $(\mathbf{EH})^t$

   (h) $|\mathbf{HE}|$

   (i) $\mathbf{EHG}$

   (j) $\mathbf{EG} - \mathbf{HG}$

   (k) $\mathbf{EH} - \mathbf{H}^t\mathbf{E}^t$

   (l) $\mathbf{F}^{-1}$

2. Suppose the x_2-axis of a rectangular coordinate system is rotated by 30° away from the x_3-axis around the x_1-axis.

(a) Find the corresponding transformation matrix. Try to do this by drawing a diagram instead of going to the book or the notes for a formula.

(b) Is this an orthogonal matrix? If so, show that it satisfies the main properties of an orthogonal matrix. If not, explain why it fails to be orthogonal.

(c) Does this matrix represent a proper or an improper rotation? How do you know?

3. When you were first introduced to vectors, you most likely were told that a scalar is a quantity that is defined by a magnitude, while a vector has both a magnitude and a direction. While this is certainly true, there is another, more sophisticated way to define a scalar quantity and a vector quantity: through their transformation properties. A scalar quantity transforms as:

$$\phi' = \phi$$

while a vector quantity transforms as follows:

$$A'_i = \sum_j \lambda_{ij} A_j$$

For example, to show that the scalar product does indeed transform as a scalar, note that:

$$\begin{aligned}
\mathbf{A}' \cdot \mathbf{B}' &= \sum_i A'_i B'_i \\
&= \sum_i \left(\sum_j \lambda_{ij} A_j \right) \left(\sum_k \lambda_{ik} B_k \right) \\
&= \sum_{j,k} \left(\sum_i \lambda_{ij} \lambda_{ik} \right) A_j B_k \\
&= \sum_j \left(\sum_k \delta_{jk} A_j B_k \right) \\
&= \sum_j A_j B_j \\
&= \mathbf{A} \cdot \mathbf{B}
\end{aligned}$$

Now you will show that the vector product transforms as a vector. Begin by writing out what you are trying to show explicitly and show it to the TA. Once the TA has confirmed that you have the correct expression, try to prove it. The vector product is a bit more difficult to work with than the scalar product, so your TA is prepared to give you a hint if you get stuck.

4. The goal of this problem is to help you understand the origin of the equations that relate two different coordinate systems. First, the TA will work through the equations relating the rectangular and cylindrical coordinate systems. Then, in small groups, you will work through the equations relating the rectangular and spherical coordinate systems.

   Refer to diagrams for cylindrical and spherical coordinates as your TA explains how to arrive at expressions for x_1, x_2, and x_3 in terms of r, ϕ, and z. Your TA will also show you how to derive expressions for the unit vectors in the cylindrical coordinate system in terms of the unit vectors in the rectangular system, as well as the time derivatives of these unit vectors. Finally, the TA will show you how to derive expressions for the velocity and acceleration vectors in cylindrical coordinates.

   After you understand how to relate the cylindrical and rectangular coordinate systems, go ahead and do the same for the spherical and rectangular coordinate systems. Your group should derive expressions relating the coordinates of the two systems, expressions relating the unit vectors and their time derivatives of the two systems, and finally, expressions for the velocity and acceleration in spherical coordinates.

5. After working on your physics problem set for a few hours, you decide to take a break and do a little hiking. You encounter a hill whose shape is given by the equation

   $$z = 980 - 0.01x^2 - 0.04y^2$$

   and you are standing at a point with coordinates (60, 100, 544).

   (a) How high is the hill?

   (b) In which direction should you proceed initially in order to reach the top of the hill fastest?

   (c) If you climb in that direction, at what angle above the horizontal will you be climbing initially?

6. Below you will find a set of integrals. Your TA will divide you into groups and each group will be assigned one integral to work on. Once your group has solved the integral, write the solution on the board in the space provided by the TA.

   (a) $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\theta} r^2 \sin\theta dr d\theta d\phi$

   (b) $\int \left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2} \right) dt$

   (c) $\int_S \mathbf{A} \cdot d\mathbf{a}$ where $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 9$.

   (d) $\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{a}$ where $\mathbf{A} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and S is the surface defined by the paraboloid $z = 1 - x^2 - y^2$, where $z \geq 0$.

7. Suppose you have two rectangular coordinate systems that share a common origin, but one system is rotated by an angle θ with respect to the other. To describe this rotation, you have made use of the rotation matrix $\lambda(\theta)$. (I'm changing the notation slightly to put the emphasis on the angle of rotation.)

   (a) Verify that the product of two rotation matrices $\lambda(\theta_1)\lambda(\theta_2)$ is in itself a rotation matrix.

   (b) In abstract algebra, a group G is defined as a set of elements g together with a binary operation $*$ acting on that set such that four properties are satisfied:

      i. (Closure) For any two elements g_i and g_j in the group G, the product of the elements, $g_i * g_j$ is also in the group G.

      ii. (Associativity) For any three elements g_i, g_j, g_k of the group G, $(g_i * g_j) * g_k = g_i * (g_j * g_k)$.

      iii. (Existence of Identity) The group G contains an identity element e such that $g * e = e * g = g$ for all $g \in G$.

      iv. (Existence of Inverses) For each element $g \in G$, there exists an inverse element $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

      Show that if the product $*$ denotes the product of two matrices, then the set of rotation matrices together with $*$ forms a group. This group is known as the special orthogonal group in two dimensions, also known as $SO(2)$.

   (c) Is this group commutative? In abstract algebra, a commutative group is called an abelian group.