Classical Mechanics - Module 4

1. Consider the use of equations of constraint.

   (a) A particle is constrained to move on the surface of a sphere. What are the equations of constraint for this system?

   (b) A disk of mass m and radius R rolls without slipping on the outside surface of a half-cylinder of radius $5R$. What are the equations of constraint for this system?

   (c) What are holonomic constraints? Which of the equations of constraint that you found above are holonomic?

   (d) Equations of constraint that do not explicitly contain time are said to be scleronomic. Moving constraints are rheonomic. Are the equations of constraint that you found above scleronomic or rheonomic?

2. For each of the following systems, describe the generalized coordinates that would work best. There may be more than one answer for each system.

   (a) An inclined plane of mass M is sliding on a smooth horizontal surface, while a particle of mass m is sliding on the smooth inclined surface.

   (b) A disk rolls without slipping across a horizontal plane. The plane of the disk remains vertical, but it is free to rotate about a vertical axis.

   (c) A double pendulum consisting of two simple pendula, with one pendulum suspended from the bob of the other. The two pendula have equal lengths and have bobs of equal mass. Both pendula are confined to move in the same plane.

   (d) A particle of mass m is constrained to move on a circle of radius R. The circle rotates in space about one point on the circle, which is fixed. The rotation takes place in the plane of the circle, with constant angular speed ω, in the absence of a gravitational force.

   (e) A particle of mass m is attracted toward a given point by a force of magnitude k/r^2, where k is a constant.

3. Looking back at the systems in problem 2, which ones could have equations of constraint? How would you classify the equations of constraint (holonomic, scleronomic, rheonomic, etc.)?

4. A disk of mass M and radius R rolls without slipping down a plane inclined from the horizontal by an angle α. The disk has a short weightless axle of negligible radius. From this axis is suspended a simple pendulum of length $l < R$ and whose bob has a mass m. Assume that the motion of the pendulum takes place in the plane of the disk.

   (a) What generalized coordinates would be appropriate for this situation?

   (b) Are there any equations of constraint? If so, what are they?

   (c) Find Lagrange's equations for this system.

5. A Lagrangian for a particular system can be written as

$$L = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2)$$

   where a, b, and c are arbitrary constants, but subject to the condition that $b^2 - 4ac \neq 0$.

   (a) What are the equations of motion?

   (b) Examine the case $a = 0 = c$. What physical system does this represent?

   (c) Examine the case $b = 0$ and $a = -c$. What physical system does this represent?

(d) Based on your answers to (b) and (c), determine the physical system represented by the Lagrangian given above.

6. Consider a particle of mass m moving in a plane and subject to an inverse square attractive force.

   (a) Obtain the equations of motion.

   (b) Is the angular momentum about the origin conserved?

   (c) Obtain expressions for the generalized forces. Recall that the generalized forces are defined by

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}.$$

7. Consider a Lagrangian function of the form $L(q_i, \dot{q}_i, \ddot{q}_i, t)$. Here the Lagrangian contains a time derivative of the generalized coordinates that is higher than the first. When working with such Lagrangians, the term "generalized mechanics" is used.

   (a) Consider a system with one degree of freedom. By applying the methods of the calculus of variations, and assuming that Hamilton's principle holds with respect to variations which keep both q and $\dot{q}$ fixed at the end points, show that the corresponding Lagrange equation is

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0.$$

   Such equations of motion have interesting applications in chaos theory.

   (b) Apply this result to the Lagrangian

$$L = -\frac{m}{2} q\ddot{q} - \frac{k}{2} q^2.$$

   Do you recognize the equations of motion?

8. A bead of mass m slides under gravity along a smooth wire bent in the shape of a parabola $x^2 = az$ in the vertical (x, z) plane.

   (a) What kind (holonomic, nonholonomic, scleronomic, rheonomic) of constraint acts on m?

   (b) Set up Lagrange's equation of motion for x with the constraint embedded.

   (c) Set up Lagrange's equations of motion for both x and z with the constraint adjoined and a Lagrangian multiplier λ introduced.

   (d) Show that the same equation of motion for x results from either of the methods used in part (b) or part (c).

   (e) Express λ in terms of x and $\dot{x}$.

   (f) What are the x and z components of the force of constraint in terms of x and $\dot{x}$?

9. Consider the two Lagrangians

$$L(q, \dot{q}; t) \quad \text{and} \quad L'(q, \dot{q}; t) = L(q, \dot{q}; t) + \frac{dF(q, t)}{dt}$$

   where $F(q, t)$ is an arbitrary function of the generalized coordinates $q(t)$. Show that these two Lagrangians yield the same Euler-Lagrange equations. As a consequence two Lagrangians that differ only by an exact time derivative are said to be equivalent.