P235 **PRACTICE FINAL EXAMINATION** **Prof Cline**

THIS IS A CLOSED BOOK EXAM. Show all steps to get full credit. Answer questions 1,2 in Book 1: questions 3, 4, in Book 2.

Book 1

1; [25pts] A point particle of mass m in a gravitational field is constrained to slide on the inner surface of a frictionless smooth paraboloid. The equation of the paraboloid is

$$x^2 + y^2 = az$$

with the z axis of the paraboloid vertical and with gravity acting vertically downward.

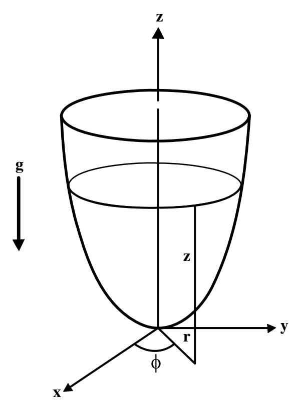
a) Find the Lagrangian for the motion in cylindrical coordinates (ρ, ϕ, z)

b) Use Noether's theorem to identify all constants of motion

c) Derive the equations of motion using the Lagrangian.

d) Derive the forces of constraint

e) Show that at any instant there is a central force directed toward a point on the z-axis on the same horizontal plane as the particle.



2; [25pts] Consider a uniform, infinitely-thin, rigid, circular disk of radius R and mass M. Assume that the body-fixed symmetry axis is the 3 axis in the body-fixed frame. The thin disk is thrown like a discus such that it spins with angular velocity ω about an axis that makes an angle α to the symmetry axis of the disk where α is a small angle. Assume that no torques or drag act on the disk.

a) Derive the principal moments of inertia about the center of mass of the disk.

b) Use the Euler equations to prove that the component of the angular velocity ω along the symmetry 3 axis, ω_3, is a constant of motion.

c) Prove that the total angular frequency ω is a constant of motion.

d) Derive the Lagrangian expressed in terms of the Euler angles and angular velocities for this system.

e) If $\mathbf{p}_\phi$ is the angular momentum conjugate to the angle ϕ about the space-fixed $\mathbf{z}$ axis, and $\mathbf{p}_\psi$ is conjugate to the angle ψ about the body-fixed $\mathbf{3}$ axis, then use the Lagrangian to prove that both $\mathbf{p}_\phi$ and $\mathbf{p}_\psi$ are constants of motion for rotation of the disk.

Book 2

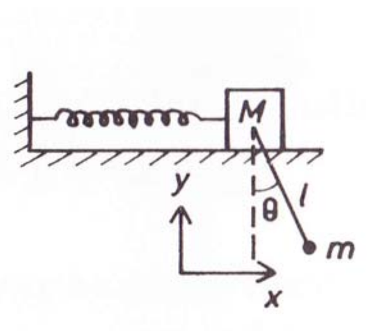
3; (25pts) A simple pendulum consists of a mass m attached to a massless string of length l which is hung from a support of mass M that slides on a frictionless horizontal rail. The mass M is attached to a fixed point via a horizontal massless spring having a force constant κ as shown in the figure.

a) Derive the Lagrangian.

b) Derive the Lagrange equations of motion

c) Find the angular frequencies of the normal modes of this system assuming small amplitude oscillations.

d) Sketch diagrams showing the relative motion of the mass M and the pendulum corresponding to each of the normal modes.



4; (25pts) A rigid straight, frictionless, massless, rod rotates about the **z** axis at an angular velocity $\dot{\theta}$. A mass m slides along the frictionless rod and is attached to the rod by a massless spring of spring constant κ.

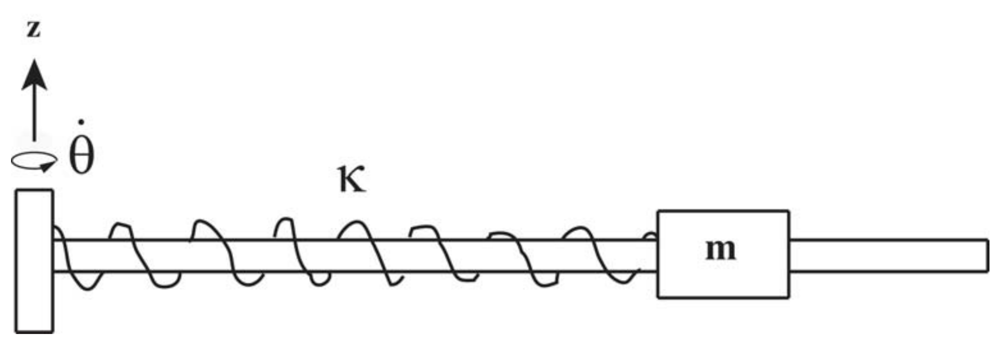
a; Derive the Lagrangian and the Hamiltonian

b; Derive the equations of motion in the stationary frame using Hamiltonian mechanics.

c; What are the constants of motion?

d; If the rotation is constrained to have a constant angular velocity $\dot{\theta} = \omega$ is the non-cyclic Routhian $R_{noncyclic} = H - p_\theta \dot{\theta}$ a constant of motion, and does it equal the total energy?

e; Use the non-cyclic Routhian $R_{noncyclic}$ to derive the radial equation of motion in the rotating frame of reference for the cranked system.



Useful formulae

Damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Solution for $\frac{\Gamma}{2} < \omega_o$ is

$$x = Ae^{-\frac{\Gamma t}{2}} \cos(\omega_1 t - \delta)$$

where

$$\omega_1^2 = \omega_0^2 - \frac{\Gamma^2}{4}$$

Sinusoidal-driven damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = A\cos(\omega t)$$

Steady-state solution is

$$x_{ss} = \frac{A}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \Gamma^2\omega^2}} \cos(\omega t - \delta)$$

where

$$\tan\delta = \frac{\Gamma\omega}{(\omega_0^2 - \omega^2)}$$

Newton's law of Gravitation

$$\begin{aligned} \overline{\mathbf{F}}_m &= -G\frac{mM}{r^2}\widehat{\mathbf{r}} \\ \overrightarrow{\boldsymbol{\nabla}} \cdot \overrightarrow{\mathbf{g}} &= -4\pi G\rho \end{aligned}$$

Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Lagrange equations with undetermined multipliers

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{k=1}^{m} \lambda_k(t)\frac{\partial g_k}{\partial q_i} \qquad (i = 1, 2, 3, ...s)$$

Generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

Hamilton's equations of motion

$$\begin{aligned} \dot{q}_j &= \frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial q_j} + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC}\right] \\ \frac{dH(\mathbf{q}, \mathbf{p}, t)}{dt} &= \sum_j \left(\left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC}\right]\dot{q}_j\right) - \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial t} \end{aligned}$$

Cylindrical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - U(\rho, z, \phi)$$

$$H = T + U = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right) + U(\rho, z, \phi)$$

Spherical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r, \theta, \phi)$$

$$H = T + U = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r, \theta, \phi)$$

Poisson Brackets

$$[F, G] \equiv \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$\begin{aligned}
[F, F] &= 0 \\
[F, G] &= -[G, F] \\
[G, F + Y] &= [G, F] + [G, Y] \\
[G, FY] &= [G, F]\,Y + F\,[G, Y] \\
0 &= [F, [G, Y]] + [G, [Y, F]] + [Y\,[F, G]]
\end{aligned}$$

$$\dot{q}_k = [q_k, H] = \frac{\partial H}{\partial p_k}$$

$$\dot{p}_k = -[p_k, H] = -\frac{\partial H}{\partial q_k}$$

Canonical transformations

$$\mathcal{H}(Q, P, t) = H(q, p, t) + \frac{\partial F}{\partial t}$$

Generating function	Generating function derivatives	Trivial special case
$F = F_1(\mathbf{q}, \mathbf{Q}, t)$	$p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i \qquad Q_i = p_i \qquad P_i = -q_i$
$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q}\cdot\mathbf{P}$	$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i \qquad Q_i = q_i \qquad P_i = p_i$
$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \mathbf{q}\cdot\mathbf{p}$	$q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i \qquad Q_i = -q_i \qquad P_i = -p_i$
$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q}\cdot\mathbf{p} - \mathbf{Q}\cdot\mathbf{P}$	$q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i \qquad Q_i = p_i \qquad P_i = -q_i$

Hamilton-Jacobi equation

$$H(q; \frac{\partial S}{\partial q}; t) + \frac{\partial S}{\partial t} = 0$$

Orbit differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u})$$

Virial Theorem

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_i \overrightarrow{\mathbf{F}}_i \cdot \overrightarrow{\mathbf{r}}_i \right\rangle$$

Effective force in rotating reference frame

$$\overrightarrow{\mathbf{F}_{eff}} = m\overrightarrow{\mathbf{a}''} = \overrightarrow{\mathbf{F}} - m\left(\overrightarrow{\mathbf{A}} + 2\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{v}''} + \overrightarrow{\boldsymbol{\omega}} \times \left(\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{r}'}\right) + \dot{\overrightarrow{\boldsymbol{\omega}}} \times \overrightarrow{\mathbf{r}'}\right)$$

Transformation from fixed to rotating frame

$$\left(\frac{d\overrightarrow{\mathbf{G}}}{dt}\right)_{fixed} = \left(\frac{d\overrightarrow{\mathbf{G}}}{dt}\right)_{rotating} + \overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{G}}$$

Inertia tensor

$$I_{ij} \equiv \sum_{\alpha}^{N} m_{\alpha}\left[\delta_{ij}\left(\sum_{k}^{3} x_{\alpha,k}^{2}\right) - x_{\alpha,i}x_{\alpha,j}\right]$$

$$I_{ij} = \int \rho\left(\mathbf{r}'\right)\left(\delta_{ij}\left(\sum_{k}^{3} x_{k}^{2}\right) - x_i x_j\right) dV$$

Angular momentum

$$\overrightarrow{\mathbf{L}} = \sum_{i}^{n} \overrightarrow{\mathbf{L}_i} = \sum_{i}^{n} \overrightarrow{\mathbf{r}_i} \times \overrightarrow{\mathbf{p}_i}$$

$$\overrightarrow{\mathbf{L}} = \{\mathbf{I}\} \cdot \overrightarrow{\boldsymbol{\omega}}$$

$$L_i = \sum_{j}^{3} I_{ij}\omega_j$$

Parallel-axis theorem

$$J_{ij} \equiv I_{ij} + M\left(a^2\delta_{ij} - a_i a_j\right)$$

Euler equations for rigid body

$$\begin{aligned}
N_1^{ext} &= I_1\dot{\omega}_1 - \left(I_2 - I_3\right)\omega_2\omega_3 \\
N_2^{ext} &= I_2\dot{\omega}_2 - \left(I_3 - I_1\right)\omega_3\omega_1 \\
N_3^{ext} &= I_3\dot{\omega}_3 - \left(I_1 - I_2\right)\omega_1\omega_2
\end{aligned}$$

Angular velocity in body-fixed frame

$$\begin{aligned}
\omega_1 &= \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\omega_2 &= \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\
\omega_3 &= \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi}
\end{aligned}$$

Angular velocity in space-fixed frame

$$\begin{aligned}
\omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\
\omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\
\omega_z &= \dot{\phi} + \dot{\psi}\cos\theta
\end{aligned}$$

Coupled oscillators

$$T=\frac{1}{2}\sum_{j,k}T_{jk}\dot{q}_j\dot{q}_k$$

$$U=\frac{1}{2}\sum_{j,k}V_{jk}q_jq_k$$

$$\sum_{j}\left(V_{jk}-\omega_r^2T_{jk}\right)a_{jr}=0$$

$$T_{jk}\equiv\sum_{\alpha}m_\alpha\sum_{i}\frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}$$

$$V_{jk}\equiv\left(\frac{\partial^2U}{\partial q_j\partial q_k}\right)_0$$

Special Relativity

$$\begin{aligned}
\mathbf{p} &= \gamma m\mathbf{u}\\
\gamma &\equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\\
E &= \gamma mc^2\\
E_0 &= mc^2\\
E &= T+E_0\\
E^2 &= p^2c^2+m^2c^4\\
L &= -mc^2\sqrt{1-\beta^2}-U\\
H &= T+U+E_0
\end{aligned}$$

Vectors;

$$\overline{\mathbf{C}}=\overline{\mathbf{A}}\times\overline{\mathbf{B}}=\begin{vmatrix}\widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

$$\overrightarrow{\mathbf{A}}\times\overrightarrow{\mathbf{A}}=0$$

$$\overrightarrow{\mathbf{A}}\cdot\left(\overrightarrow{\mathbf{A}}\times\overrightarrow{\mathbf{B}}\right)=0$$

$$\overrightarrow{\mathbf{A}}\cdot\left(\overrightarrow{\mathbf{B}}\times\overrightarrow{\mathbf{C}}\right)=\left(\overrightarrow{\mathbf{A}}\times\overrightarrow{\mathbf{B}}\right)\cdot\overrightarrow{\mathbf{C}}$$

$$\overrightarrow{\mathbf{A}}\times\left(\overrightarrow{\mathbf{B}}\times\overrightarrow{\mathbf{C}}\right)=\overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{A}}\cdot\overrightarrow{\mathbf{C}}\right)-\left(\overrightarrow{\mathbf{A}}\cdot\overrightarrow{\mathbf{B}}\right)\overrightarrow{\mathbf{C}}$$