P235 **PRACTICE MIDTERM EXAMINATION** **Prof Cline**

Exam given 1900 - 2200 hr, 30 October 2014

THIS IS A CLOSED BOOK EXAM. Show all steps to get full credit. Answer questions 1,2 in Book 1: questions 3, 4, in Book 2.

Book 1

1:(20pts) Consider a rocket fired vertically upwards from the ground in a uniform vertical gravitation field $g = 9.81 m/s^2$. The rocket, with initial mass m_I, ejects uniformly α kg/sec of propellant at an exhaust velocity u m/sec.

a) Derive the equation giving the time dependence of the vertical velocity of the rocket

b) If the initial to final mass ratio of the rocket is $\frac{m_I}{m} = 10$, the exhaust velocity $u = 4500 m/s$ and the fuel burns at a constant rate α for 300 seconds, calculate the maximum velocity of the rocket.

c) If the rocket burns half the fuel in 150 seconds, then ejects the empty fuel tank of mass $0.05 m_I$ and then burns the remaining half of the fuel in the second 150 seconds, calculate the maximum velocity of the rocket.

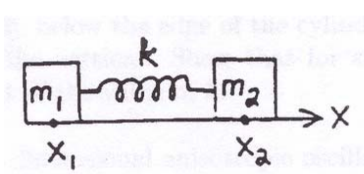
2;(25pts) Two masses m_1 and m_2 slide freely on a horizontal frictionless plane surface and are connected by a spring whose force constant is k and unstretched length is l.

a) Derive the equations of motion

b) Derive the frequency of oscillation

c) Show that the system oscillation can be represented by motion of a single equivalent mass.

d) Which observables are conserved?



Book 2

3; (30pts) A mass m_1 sliding freely on a horizontal frictionless plane, is attached by a string of length b to a second mass m_2 that hangs through a hole in the frictionless plane. Assume that mass m_1 is rotating about the hole with an angular velocity $\frac{d\phi}{dt} = \omega$ at a distance s from the hole while mass m_2 is swinging in a vertical plane at an instantaneous angle θ to the vertical at a distance r from the hole. That is, $r + s = b$.

a) Derive the Lagrangian for the system

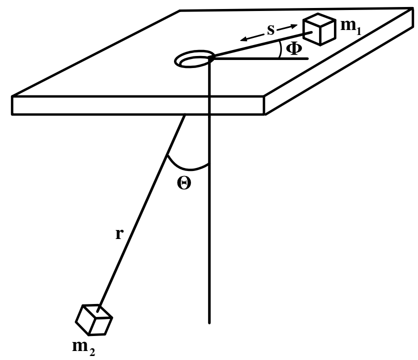
b) Use the Lagrangian to derive the equations of motion of the system for r, θ, and ϕ.

c) Derive the Hamiltonian

d) Determine all the constants of motion

e) For the simple case where $\dot{\theta} = \theta = 0$, show that there is a distance r_0 for which the system is stable to external perturbations in r.

f) Determine the angular frequency for oscillations about this stable position.



Mass m_2, hanging from a rope that is connected to m_1, which slides on a frictionless plane.

4; (25pts) A rigid straight, frictionless, massless, rod rotates about the **z** axis at an angular velocity $\dot{\theta}$. A mass m slides along the frictionless rod and is attached to the rod by a massless spring of spring constant κ and an unstretched length d

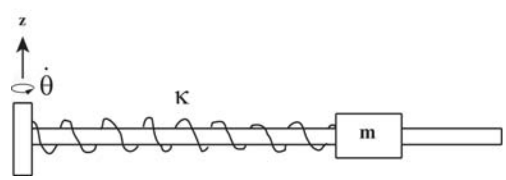
a; Derive the Lagrangian and the Hamiltonian

b; Derive the equations of motion in the stationary frame using Hamiltonian mechanics.

c; What are the constants of motion?

d; If the rotation is constrained to have a constant angular velocity $\dot{\theta} = \omega$ then is the non-cyclic Routhian $R_{noncyclic} = H - p_\theta \dot{\theta}$ a constant of motion, and does it equal the total energy?

e; Use the non-cyclic Routhian $R_{noncyclic}$ to derive the radial equation of motion in the rotating frame of reference for the cranked system with $\dot{\theta} = \omega$.



Useful formulae

Damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Solution for $\frac{\Gamma}{2} < \omega_o$ is

$$x = Ae^{-\frac{\Gamma t}{2}}\cos(\omega_1 t - \delta)$$

where

$$\omega_1^2 = \omega_0^2 - \frac{\Gamma^2}{4}$$

Sinusoidal-driven damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = A\cos(\omega t)$$

Steady-state solution is

$$x_{ss} = \frac{A}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \Gamma^2\omega^2}}\cos(\omega t - \delta)$$

$$\tan\delta = \frac{\Gamma\omega}{\left(\omega_0^2 - \omega^2\right)}$$

Newton's law of Gravitation

$$\begin{aligned} \overline{\mathbf{F}}_m &= -G\frac{mM}{r^2}\widehat{\mathbf{r}} \\ \overrightarrow{\boldsymbol{\nabla}} \cdot \overrightarrow{\mathbf{g}} &= -4\pi G\rho \end{aligned}$$

Virial Theorem

$$\langle T\rangle = -\frac{1}{2}\left\langle \sum_i \overrightarrow{\mathbf{F}}_i \cdot \overrightarrow{\mathbf{r}}_i \right\rangle$$

Lagrange equations:
Lagrange equations including undetermined multipliers and generalized forces

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{EXC} + \sum_{k=1}^{m}\lambda_k(t)\frac{\partial g_k}{\partial q_j} \qquad (i = 1, 2, 3, ...s)$$

Generalized momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Hamiltonian

$$H(q_i, p_i, t) = \sum_i p_i\dot{q}_i - L(q_i, \dot{q}_i, t)$$

Hamilton's equations of motion

$$\begin{aligned}\dot{q}_k &= \frac{\partial H}{\partial p_k} \\ \dot{p}_j &= -\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial q_j} + \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC}\right] \\ \frac{dH(\mathbf{q},\mathbf{p},t)}{dt} &= \sum_j \left(\left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} + Q_j^{EXC}\right]\dot{q}_j\right) - \frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t}\end{aligned}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Routhian reduction

$$\begin{aligned}R_{cyclic}(q_1,...,q_n;\dot{q}_1,...,\dot{q}_s;p_{s+1},....,p_n;t) &= \sum_{cyclic}^{m} p_i\dot{q}_i - L = H - \sum_{noncyclic}^{s} p_i\dot{q}_i \\ R_{noncyclic}(q_1,...,q_n;p_1,...,p_s;\dot{q}_{s+1},....,\dot{q}_n;t) &= \sum_{noncyclic}^{s} p_i\dot{q}_i - L = H - \sum_{cyclic}^{m} p_i\dot{q}_i\end{aligned}$$

Cylindrical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - U(\rho z \phi)$$

$$H = T + U = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right) + U(\rho, z, \phi)$$

Spherical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r\theta\phi)$$

$$H = T + U = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r, \theta, \phi)$$

Vectors;

$$\overline{\mathbf{C}}=\overline{\mathbf{A}}\times\overline{\mathbf{B}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{A}} = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) \cdot \overrightarrow{\mathbf{C}}$$

$$\overrightarrow{\mathbf{A}} \times \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{C}}\right) - \left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}}\right)\overrightarrow{\mathbf{C}}$$