P235 - PROBLEM SET 11

To be handed in by **1800 hr on Friday, 4 December 2015**.

1. Consider the motion of a particle of mass m in an isotropic harmonic oscillator potential $U = \frac{1}{2}kr^2$and take the orbital plane to be the $x - y$ plane. The Hamiltonian is then

$$H \equiv S_0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}k(x^2 + y^2)$$

Define the three energies

$$\begin{aligned} S_1 &= \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}k(x^2 - y^2) \\ S_2 &= \frac{1}{m}p_x p_y + kxy \\ S_3 &= \omega(xp_y - yp_x) \end{aligned}$$

with $\omega = \sqrt{\frac{k}{m}}$. Use Poisson brackets to solve the following:
a) Show that $[S_0, S_i] = 0$ for $i = 1, 2, 3$ proving that (S_1, S_2, S_3) are constants of motion.
b) Show that

$$\begin{aligned} [S_1, S_2] &= 2\omega S_3 \\ [S_2, S_3] &= 2\omega S_1 \\ [S_3, S_1] &= 2\omega S_2 \end{aligned}$$

so that $(2\omega)^{-1}(S_1, S_2, S_3)$ have the same Poisson bracket relations as the components of a 3-dimensional angular momentum.
c) Show that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

2. Assume that the transformation equations between the two sets of coordinates (q, p) and (Q, P) are

$$\begin{aligned} Q &= \ln\left(\frac{\sin p}{q}\right) \\ P &= q \cot p \end{aligned}$$

a) Assuming that q, p are canonical variables, i.e. $[q, p] = 1$, show directly from the above transformation equations that Q, P are canonical variables.
b) Show that

$$pdq - PdQ = d(pq + q\cot p)$$

c) Find the explicit generating function $F_1(q, Q)$ that generates this transformation between these two sets of canonical variables. Note the integral $\int \sin^{-1} x dx = \sqrt{1 - x^2} + x \sin^{-1} x$

3. Consider a bound two-body system comprising a mass m in an orbit at a distance r from a mass M. The attractive central force binding the two-body system is

$$\mathbf{F} = \frac{k}{r^2}\hat{\mathbf{r}}$$

where k is negative. Use Poisson brackets to prove that the eccentricity vector $A = p \times L + \mu k \hat{r}$ is a conserved quantity.

4 (a) Consider the case of a single mass m where the Hamiltonian $H = \frac{1}{2}p^2$. Use the generating function $S(q, P, t)$ to solve the Hamilton-Jacobi equation with the canonical transformation $q = q(Q, P)$ and $p = p(Q, P)$ and determine the equations relating the (q, p) variables to the transformed coordinate and momentum (Q, P).

(b) If there is a perturbing Hamiltonian $\Delta H = \frac{1}{2}q^2$, then P will not be constant. Express the transformed Hamiltonian $\mathcal{H}$ (using the transformation given above in terms of P, Q, and t). Solve for $Q(t)$ and $P(t)$ and show that the perturbed solution $q[Q(t), P(t)]$, $p[Q(t), P(t)]$ is simple harmonic.