1. Consider the motion of a particle of mass $m$ in an isotropic harmonic oscillator potential $U = \frac{1}{2}kr^2$ and take the orbital plane to be the $x-y$ plane. The Hamiltonian is then

$$H \equiv S_0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}k(x^2 + y^2)$$

Define the three energies

$$S_1 = \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}k(x^2 - y^2)$$
$$S_2 = \frac{1}{m}pxpy + kxy$$
$$S_3 = \omega(xpy - ypx)$$

with $\omega = \sqrt{\frac{k}{m}}$. Use Poisson brackets to solve the following:

a) Show that $[S_0, S_i] = 0$ for $i = 1, 2, 3$ proving that $(S_1, S_2, S_3)$ are constants of motion.

b) Show that

$$[S_1, S_2] = 2\omega S_3$$
$$[S_2, S_3] = 2\omega S_1$$
$$[S_3, S_1] = 2\omega S_2$$

so that $(2\omega)^{-1}(S_1, S_2, S_3)$ have the same Poisson bracket relations as the components of a 3-dimensional angular momentum.

c) Show that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

2. Assume that the transformation equations between the two sets of coordinates $(q, p)$ and $(Q, P)$ are

$$Q = \ln\left(\frac{\sin p}{q}\right)$$
$$P = q\cot p$$

a) Assuming that $q, p$ are canonical variables, i.e. $[q, p] = 1$, show directly from the above transformation equations that $Q, P$ are canonical variables.

b) Show that

$$pdp - Pdq = d(pq + q\cot p)$$

c) Find the explicit generating function $F_1(q, Q)$ that generates this transformation between these two sets of canonical variables. Note the integral

$$\int \sin^{-1}xdx = \sqrt{1-x^2} + x\sin^{-1}x$$

3. Consider a bound two-body system comprising a mass $m$ in an orbit at a distance $r$ from a mass $M$. The attractive central force binding the two-body system is

$$\mathbf{F} = \frac{k}{r^2}\hat{r}$$

where $k$ is negative. Use Poisson brackets to prove that the eccentricity vector $A = p \times L + \mu k\hat{r}$ is a conserved quantity.