P235 - PROBLEM SET 3

To be handed in by 1800 hr on Friday, 25 September 2015.

1. Find the extremal of the functional
   \[ J(x) = \int_0^\pi (2x \sin t - \dot{x}^2) dt \]
   that satisfies \( x(\alpha) = x(\pi) = 0 \). Show that this extremal provides the global maximum of \( J \).

2. Find and describe the path \( y = y(x) \) for which the integral \( \int_{x_1}^{x_2} \sqrt{\frac{x}{1 + (y')^2}} dx \) is stationary.

3. Find the dimensions of the parallelepiped of maximum volume circumscribed by a sphere of radius \( R \).

4. Consider a single loop of the cycloid having a fixed value of \( a \) as shown in the figure. A car released from
   rest at any point \( P_0 \) anywhere on the track between \( O \) and the lowest point \( P \), that is, \( P_0 \) has a parameter
   \( 0 < \theta_0 < \pi \).

   \[ \begin{align*}
   (a) \text{ Show that the time } T \text{ for the cart to slide from } P_0 \text{ to } P \text{ is given by the integral }
   \\
   T(P_0 \to P) = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos \theta}{\cos \theta_0 - \cos \theta}} d\theta
   \\
   \end{align*} \]

   (b) Prove that this time \( T \) is equal to \( \pi \sqrt{a/g} \) which is independent of the position \( P_0 \).

   (c) Explain qualitatively how this surprising result can possibly be true.

5. Consider a medium for which the refractive index \( n = \frac{a}{r} \) where \( a \) is a constant and \( r \) is the distance from
   the origin. Use Fermat’s Principle to find the path of a ray of light travelling in a plane containing the origin.
   Hint, use two-dimensional polar coordinates with \( \phi = \phi(r) \). Show that the resulting path is a circle through
   the origin.

6. Find the shortest path between the \((x, y, z)\) points \((0, -1, 0)\) and \((0, 1, 0)\) on the conical surface
   \[ z = 1 - \sqrt{x^2 + y^2} \]
   What is the length of this path? Note that this is the shortest mountain path around a volcano.

7. Show that the geodesic on the surface of a right circular cylinder is a segment of a helix.