P235 - PROBLEM SET 3

To be handed in by **1700 hr on Friday, 24 September 2010**.

[1] An unusual pendulum is made by fixing a string to a horizontal cylinder of radius R,wrapping the string several times around the cylinder, and then tying a mass m to the loose end. In equilibrium the mass hangs a distance l_0 vertically below the edge of the cylinder. Find the potential energy if the pendulum has swung to an angle ϕ from the vertical. Show that for small angles, it can be written in the Hooke's Law form $U = \frac{1}{2}k\phi^2$. Comment of the value of k.

[2] Consider the two-dimensional anisotropic oscillator with motion with $\omega_x = p\omega$ and $\omega_y = q\omega$.

a) Prove that if the ratio of the frequencies is rational (that is, $\frac{\omega_x}{\omega_y} = \frac{p}{q}$ where p and q are integers) then the motion is periodic. What is the period?

b) Prove that if the same ratio is irrational, the motion never repeats itself.

[3] A simple pendulum consists of a mass m suspended from a fixed point by a weight-less, extensionless rod of length l.

a) Obtain the equation of motion, and in the approximation $\sin\theta \approx \theta$, show that the natural frequency is $\omega_0 = \sqrt{\frac{g}{l}}$, where g is the gravitational field strength.

b) Discuss the motion in the event that the motion takes place in a viscous medium with retarding force $2m\sqrt{gl}\dot{\theta}$.

[4] Derive the expression for the State Space paths of the plane pendulum if the total energy is $E > 2mgl$. Note that this is just the case of a particle moving in a periodic potential $U(\theta) = mgl(1-\cos\theta)$. Sketch the State Space diagram for both $E > 2mgl$ and $E < 2mgl$.

[5] Consider the motion of a driven linearly-damped harmonic oscillator after the transient solution has died out, and suppose that it is being driven close to resonance, $\omega = \omega_o$.

a) Show that the oscillator's total energy is $E = \frac{1}{2}m\omega^2 A^2$.

b) Show that the energy ΔE_{dis} dissipated during one cycle by the damping force $\Gamma\dot{x}$ is $\pi\Gamma m\omega A^2$

[6] Two masses m_1 and m_2 slide freely on a horizontal frictionless surface and are connected by a spring whose force constant is k. Find the frequency of oscillatory motion for this system.

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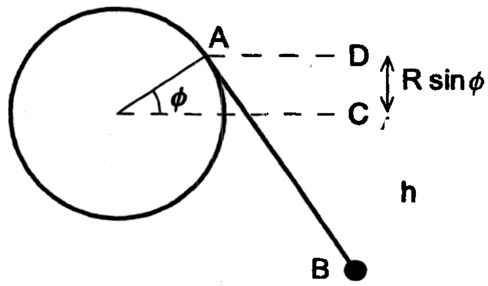
[1] An unusual pendulum is made by fixing a string to a horizontal cylinder of radius R,wrapping the string several times around the cylinder, and then tying a mass m to the loose end. In equilibrium the mass hangs a distance l_0 vertically below the edge of the cylinder. Find the potential energy if the pendulum has swung to an angle ϕ from the vertical. Show that for small angles, it can be written in the Hooke's Law form $U = \frac{1}{2}k\phi^2$. Comment of the value of k.



The PE is $U = -mgh$ where h is the height of the mass, measured down from the level of the cylinder's center. To find h, note first that as the pendulum swings from equilibrium to angle ϕ, a length $R\phi$ of string unwinds from the cylinder. Thus the length of string away from the cylinder is $AB = (l_o + R\phi)$, and the height BD is $BD = (l_o + R\phi)\cos\phi$. Since the height $CD = R\sin\phi$, we find by subtraction that $h = BD - CD = l_o\cos\phi + R(\phi\cos\phi - \sin\phi)$. Therefore

$$U = -mgh = -mg[l_o\cos\phi + R(\phi\cos\phi - \sin\phi)].$$

If ϕ remains small we can write $\cos\phi \approx 1 - \phi^2/2$ and $\sin\phi \approx \phi$, to give

$$U \approx -mg\left\{l_o - \tfrac{1}{2}l_o\phi^2 + R\left[\phi(1 - \tfrac{1}{2}\phi^2) - \phi\right]\right\} \approx -mgl_o + \tfrac{1}{2}mgl_o\phi^2 = \text{const} + \tfrac{1}{2}k\phi^2$$

where in the third expression I dropped the term in ϕ^3. The constant $k = mgl_o$, which is the same as for a simple pendulum of length l_o. Evidently, wrapping the string around a cylinder makes no difference for small oscillations.

[2] Consider the two-dimensional anisotropic oscillator with motion with $\omega_x = p\omega$ and $\omega_y = q\omega$.

a) Prove that if the ratio of the frequencies is rational (that is, $\frac{\omega_x}{\omega_y} = \frac{p}{q}$ where p and q are integers) then the motion is periodic. What is the period?

b) Prove that if the same ratio is irrational, the motion never repeats itself.

(a) Suppose that the ratio of frequencies is rational, that is $\omega_x/\omega_y = p/q$, where p and q are integers. Then let $\tau = 2\pi p/\omega_x = 2\pi q/\omega_y$. Now consider the following

$$x(t+\tau) = A_x \cos[\omega_x(t+\tau]) = A_x \cos[\omega_x t + 2\pi p] = A_x \cos[\omega_x t] = x(t)$$

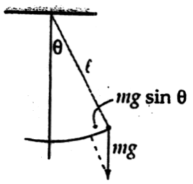
where in the second equality I used our definition of τ and in the second the fact that if p is an integer then $\cos(\theta + 2\pi p) = \cos(\theta)$. This shows that $x(t)$ is periodic with period τ. By exactly the same argument, $y(t)$ is also periodic with the same period τ, and we've proved that the whole motion is likewise. What we usually call *the* period of the motion is the value of $\tau = 2\pi p/\omega_x$ with p and q the *smallest* integers for which $\omega_x/\omega_y = p/q$.

(b) Suppose the motion is periodic. Then there is a τ such that $x(t+\tau) = x(t)$ and $y(t+\tau) = y(t)$. Running the previous argument backward, we see that $\omega_x\tau$ must be an integer multiple of 2π, that is $\omega_x\tau = 2\pi p$ for some integer p. Similarly $\omega_y\tau = 2\pi q$ for some integer q. Dividing these two conclusions, we see that $\omega_x/\omega_y = p/q$ and the ratio of frequencies is rational. Therefore, if the ratio is irrational, the motion cannot be periodic.

3] A simple pendulum consists of a mass m suspended from a fixed point by a weight-less, extensionless rod of length l.

a) Obtain the equation of motion, and in the approximation $\sin\theta \approx \theta$, show that the natural frequency is $\omega_0 = \sqrt{\frac{g}{l}}$, where g is the gravitational field strength.

b) Discuss the motion in the event that the motion takes place in a viscous medium with retarding force $2m\sqrt{gl}\dot{\theta}$.



The equation of motion is

$$-m\ell\ddot{\theta} = mg\sin\theta \tag{1}$$

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{2}$$

If θ is sufficiently small, we can approximate $\sin\theta \cong \theta$, and (2) becomes

$$\ddot{\theta} = -\frac{g}{\ell}\theta \tag{3}$$

which has the oscillatory solution

$$\theta(t) = \theta_0 \cos\omega_0 t \tag{4}$$

where $\omega_0 = \sqrt{g/\ell}$ and where θ_0 is the amplitude. If there is the retarding force $2m\sqrt{g\ell}\,\dot{\theta}$, the equation of motion becomes

$$-m\ell\ddot{\theta} = mg\sin\theta + 2m\sqrt{g\ell}\,\dot{\theta} \tag{5}$$

or setting $\sin\theta \cong \theta$ and rewriting, we have

$$\ddot{\theta} + 2\omega_0\dot{\theta} + \omega_0^2\theta = 0 \tag{6}$$

Comparing this equation with the standard equation for damped motion [Eq. (3.35)],

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{7}$$

we identify $\omega_0 = \beta$. This is just the case of *critical damping*, so the solution for $\theta(t)$ is [see Eq. (3.43)]

$$\theta(t) = (A + Bt)e^{-\omega_0 t} \tag{8}$$

For the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, we find

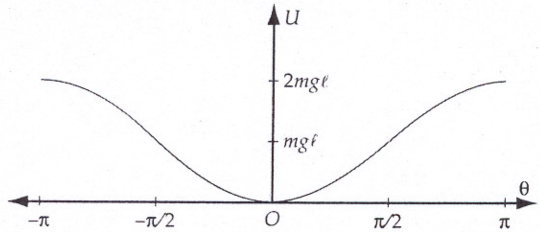
$$\boxed{\theta(t) = \theta_0(1 + \omega_0 t)e^{-\omega_0 t}}$$

[4] Derive the expression for the State Space paths of the plane pendulum if the total energy is $E > 2mgl$. Note that this is just the case of a particle moving in a periodic potential $U(\theta) = mgl(1-\cos\theta)$. Sketch the State Space diagram for both $E > 2mgl$ and $E < 2mgl$.

For the plane pendulum, the potential energy is

$$u = mg\ell\left[1-\cos\theta\right] \tag{1}$$

If the total energy is *larger* than $2mg\ell$, all values of θ are allowed, and the pendulum revolves continuously in a circular path. The potential energy as a function of θ is shown in (a) below.



(a)

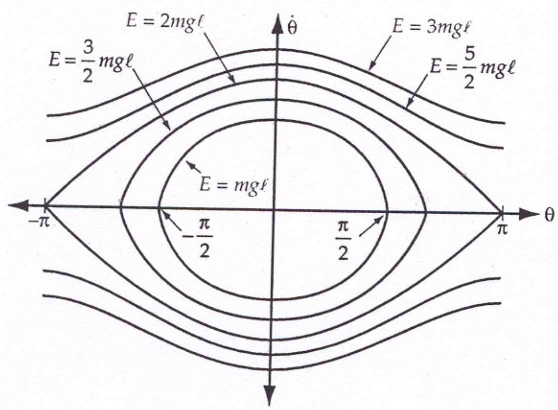
Since $T = E - U(\theta)$, we can write

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\dot{\theta}^2 = E - mg\ell\left(1-\cos\theta\right) \tag{2}$$

and, therefore, the phase paths are constructed by plotting

$$\dot{\theta} = \sqrt{\frac{2}{m\ell^2}}\left[E - mg\ell\left(1-\cos\theta\right)\right]^{1/2} \tag{3}$$

versus θ. The phase diagram is shown in (b) below.



[5] Consider the motion of a driven linearly-damped harmonic oscillator after the transient solution has died out, and suppose that it is being driven close to resonance, $\omega = \omega_o$.

a) Show that the oscillator's total kinetic energy is $E = \frac{1}{2}m\omega^2 A^2$.

b) Show that the energy ΔE_{dis} dissipated during one cycle by the damping force $\Gamma\dot{x}$ is $\pi\Gamma m\omega A^2$

(a) Since $x = A\cos(\omega t - \delta)$, the total energy is

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2 A^2\cos^2(\omega t - \delta) + \tfrac{1}{2}kA^2\sin^2(\omega t - \delta).$$

Because $\omega \approx \omega_o$, we can replace $k = m\omega_o^2$ by $m\omega^2$, and then, since $\cos^2\theta + \sin^2\theta = 1$, we get $E = \frac{1}{2}m\omega^2 A^2$, as claimed.

(b) The rate at which the damping force dissipates energy is $F_{\text{dmp}}v = bv^2 = 2m\beta v^2$. Therefore the energy dissipated in one period is

$$\Delta E_{\text{dis}} = \int_0^T 2m\beta v^2 dt = 2m\beta\omega^2 A^2 \int_0^T \sin^2(\omega t - \delta)dt.$$

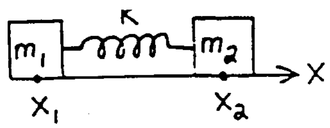
The remaining integral is just π/ω. (To see this use the trig identity $\sin^2\theta = \frac{1}{2}(1 - \sin 2\theta)$ and note that the integral of the sine term over a period is zero.) Therefore, $\Delta E_{\text{dis}} = 2\pi m\beta\omega A^2$.

(c) Combining the results of parts (a) and (b), we find that

$$\frac{E}{\Delta E_{\text{dis}}} = \frac{\frac{1}{2}m\omega^2 A^2}{2\pi m\beta\omega A^2} = \frac{\omega_o}{4\pi\beta} = \frac{Q}{2\pi}$$

where I have again used the fact that $\omega = \omega_o$. That is, the ratio of the total energy to the energy lost per cycle is $2\pi Q$.

[6] Two masses m_1 and m_2 slide freely on a horizontal frictionless surface and are connected by a spring whose force constant is k. Find the frequency of oscillatory motion for this system.



Suppose the coordinates of m_1 and m_2 are x_1 and x_2 and the length of the spring at equilibrium is ℓ. Then the equations of motion for m_1 and m_2 are

$$m_1\ddot{x}_1 = -k(x_1-x_2+\ell) \tag{1}$$

$$m_2\ddot{x}_2 = -k(x_2-x_1-\ell) \tag{2}$$

From (2), we have

$$x_1 = \frac{1}{k}(m_2\ddot{x}_2 + kx_2 - k\ell) \tag{3}$$

Substituting this expression into (1), we find

$$\frac{d^2}{dt^2}[m_1m_2\ddot{x}_2 + (m_1+m_2)kx_2] = 0 \tag{4}$$

from which

$$\ddot{x}_2 = -\frac{m_1+m_2}{m_1m_2}kx_2 \tag{5}$$

Therefore, x_2 oscillates with the frequency

$$\boxed{\omega = \sqrt{\frac{m_1+m_2}{m_1m_2}k}} \tag{6}$$

We obtain the same result for x_1. If we notice that the reduced mass of the system is defined as [see Eq. (7.5)]

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{7}$$

we can rewrite (6) as

$$\omega = \sqrt{\frac{k}{\mu}} \tag{8}$$

This means the system oscillates in the same way as a system consisting of a single mass μ.