P235 Midterm Examination Prof. Cline

7:00-10.00pm, 30 October 2009

THIS IS A CLOSED BOOK EXAMINATION.

Do all parts of all four questions.

Show all steps to get full credit.

1:(20pts) Consider a rocket fired vertically upwards from the ground in a uniform vertical gravitation field $g = 9.81m/s^2$. The rocket, with initial mass m_I, ejects α kg/sec of propellant at an exhaust velocity u m/sec. Derive the equation giving the time dependence of the vertical velocity of the rocket.

SOLUTION:

When there is a vertical gravitational external field the vertical momentum is not conserved. In a time dt the rocket ejects propellant dm_p with exhaust velocity u. Thus the momentum imparted to this propellant is

$$dp_p = -udm_p$$

Therefore the rocket is given an equal and opposite increase in momentum

$$dp_R = +udm_p$$

The thrust F due to ejection of the exhaust is given by

$$\overline{\mathbf{F}} = \frac{d\overline{\mathbf{p}_R}}{dt} = -\frac{d\overline{\mathbf{p}_p}}{dt}$$

Consider the total problem for the special case of vertical ascent of the rocket against a gravitational force $F_G = -mg$. Then

$$-mg + u\frac{dm_p}{dt} = m\frac{dv_R}{dt}$$

This can be rewritten as

$$-mg + u\dot{m}_p = m\dot{v}_R$$

The second term comes from the variable mass. Let the fuel burn be constant at

$$\dot{m} = -\dot{m}_p = -\alpha$$

where $\alpha > 0$. Then the equation becomes

$$dv = \left(-g + \frac{\alpha}{m}u\right) dt$$

Since

$$\frac{dm}{dt} = -\alpha$$

then

$$-\frac{dm}{\alpha} = dt$$

Inserting this in the above equation gives

$$dv = \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm$$

Integration gives

$$v = -\frac{g}{\alpha}(m_I - m) + u\ln\left(\frac{m_I}{m}\right)$$

But the change in mass is given by

$$\int_{m_I}^{m} dm = -\alpha\int_0^t dt$$

That is

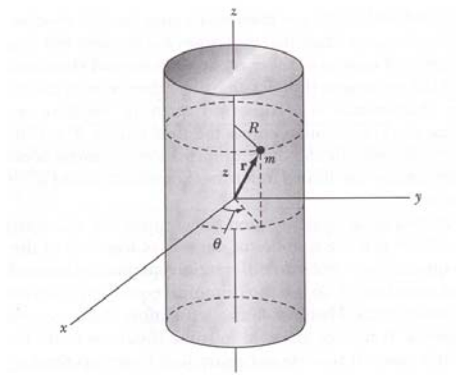
$$m_I - m = \alpha t$$

Thus

$$v = -gt + u\ln\left(\frac{m_I}{m}\right)$$

2; [30pts] Consider a mass m attracted by a force that is directed toward the origin and proportional to the distance from the origin, $\bar{\mathbf{F}} = -k\bar{\mathbf{r}}$. The mass is constrained to move on the surface of a cylinder with the origin on the cylindrical axis as shown. The radius of the cylinder is R.

a) Derive the Lagrangian and Hamiltonian
b) Explain if the Hamiltonian equals the total energy and if it is conserved?
c) Derive the equations of motion
d) Show which variables, if any, are cyclic
e) Derive the frequency of the motion along the axis of the cylinder.



SOLUTION

This problem considers the case where a mass m is attracted by a force directed toward the origin and proportional to the distance from the origin. and the mass is constrained to move on the surface of a cylinder defined by

$$x^2 + y^2 = R^2$$

a) Derive the Lagrangian and Hamiltonian

It is obvious to use cylindrical coordinates ρ, z, θ for this problem. Since the force is just Hooke's law

$$\overrightarrow{\mathbf{F}} = -k\overrightarrow{\mathbf{r}}$$

the potential is the same as for the harmonic oscillator, that is

$$U = \frac{1}{2}kr^2 = \frac{1}{2}k(\rho^2 + z^2)$$

That is, it is independent of θ.

In cylindrical coordinates the velocity is

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$$

Confined to the surface of the cylinder means that

$$\begin{aligned}\rho &= R \\ \dot{\rho} &= 0\end{aligned}$$

Then the Lagrangian simplifies to

$$\begin{aligned}L &= T - U \\ &= \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2}k(R^2 + z^2)\end{aligned}$$

The generalized coordinates are θ, z and the generalized momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \tag{a}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \tag{b}$$

$$\begin{aligned} H &= T + U \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(R^2 + z^2) \end{aligned}$$

b) Explain if the Hamiltonian equals the total energy and if it is conserved?

Since the $\frac{\partial L}{\partial t} = 0$ the Hamiltonian is conservative, and since the transformation from rectangular to cylindrical coordinates does not depend explicitly on time, then the Hamiltonian equals the total energy.

c) Derive the equations of motion.

This can be done directly for the Lagrange equations. That is,

$$\Lambda_z L = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$

gives

$$\ddot{z} + \frac{k}{m} z = 0$$

Similarly

$$\Lambda_\theta L = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

gives

$$\frac{d}{dt} mR^2\dot{\theta} = 0$$

d) The cyclic variables

The angular momentum about the axis of the cylinder is conserved. since $p_\theta =$ constant. That is, θ is a cyclic variable.

e)The frequency of small oscillations

$$\ddot{z} + \frac{k}{m} z = 0$$

is the equation for simple harmonic motion with angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

It is interesting that this problem has the same solutions for the z coordinate as the harmonic oscillator while the θ coordinate moves with constant angular velocity.

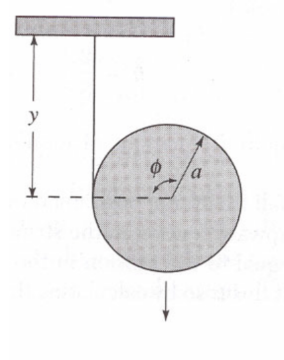


Figure 1: The yo-yo comprises a falling disc unrolling from an attached string attached to a fixed support.

3; (30pts) As part of the Halloween, and the post midterm exam celebrations, Julieta, and Wendi, organize their colleagues to drape crepe ribbons from the highest bridge in Wilson Commons. They plan to do this by dropping rolls of crepe ribbon after fastening the loose end to the parapet. Being very careful physicists they decide that first they need to know the tension in the ribbon to ensure that it does not tear when the roll drops, as well as the speed of the roll after falling $20m$ to ensure it does not hurt anyone. Assume that the rolls of ribbon have a mass M and radius a and that the thickness and mass of the unrolled ribbon is negligible compared with the mass of the roll, that is M and a are constant as it drops, thus the moment of inertia of the roll is $I = \frac{1}{2}Ma^2$.

a) Derive the Lagrangian for one roll of crepe dropping with the upper loose end of the crepe ribbon held fixed.
b) Use Lagrange multipliers to derive the equations of motion
c) Derive the tension in the ribbon.
d) Derive the vertical acceleration and final velocity of the roll after falling $20m$ assuming it starts with zero velocity.

SOLUTION:

a) Consider the roll of ribbon comprising a disc that has a ribbon wrapped around it with one end attached to a fixed support. The roll is allowed to fall with the ribbon unwinding as it falls as shown in the figure. This is identical to the yo-yo problem. Let us derive the equations of motion and the forces of constraint.

The kinetic energy of the falling yo-yo is given by

$$\begin{aligned} T &= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2 \\ &= \frac{1}{2}m\dot{y}^2 + \frac{1}{4}ma^2\dot{\phi}^2 \end{aligned}$$

where m is the mass of the disc, a the radius, and $I = \frac{1}{2}ma^2$ is the moment of inertia of the disc about its central axis. The potential energy of the disc is

$$U = -mgy$$

Thus the Lagrangian is

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}ma^2\dot{\phi}^2 + mgy$$

b) The one equation of constraint is

$$g(y,\phi) = y - a\phi = 0$$

The two Lagrange equations are

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} + \lambda\frac{\partial g}{\partial y} &= 0 \\ \frac{\partial f}{\partial \phi} - \frac{d}{dt}\frac{\partial f}{\partial \phi'} + \lambda\frac{\partial g}{\partial \phi} &= 0 \end{aligned}$$

with only one Lagrange multiplier. Evaluating these gives the two equations of motion

$$\begin{aligned} mg - m\ddot{y} + \lambda &= 0 \\ -\frac{1}{2}ma^2\ddot{\phi} - \lambda a &= 0 \end{aligned}$$

Differentiating the equation of constraint gives

$$\ddot{\phi} = \frac{\ddot{y}}{a}$$

Inserting this into the second equation and solving the two equations gives

$$\lambda = -\frac{1}{3}mg$$

Inserting λ into the two equations of motion gives

$$\begin{aligned}\ddot{y} &= \frac{2}{3}g \\ \ddot{\phi} &= \frac{2}{3}\frac{g}{a}\end{aligned}$$

c) The generalized force of constraint is the tension in the ribbon

$$F_y = \lambda\frac{\partial g}{\partial y} = -\frac{1}{3}mg$$

and the constraint torque is

$$N_\phi = \lambda\frac{\partial g}{\partial \phi} = \frac{1}{3}mga$$

d) Thus we have that the ribbon reduces the acceleration of the disc in the gravitational field by one third. $\ddot{y} = \frac{2}{3}g$
Since the acceleration is uniform and initial velocity is zero, then the final velocity is given by

$$v^2 = 2ad = \frac{4}{3}gd$$

Inserting numbers gives $v = \sqrt{1.333 \times 9.81 \times 20} = 16.17m/s$

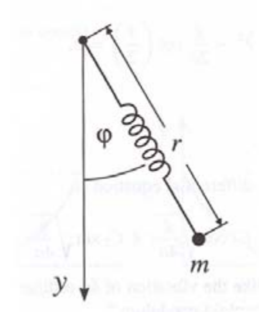


Figure 2: Spring pendulum having spring constant k and oscillating in a vertical plane.

4; [30pts] The P235W midterm exam induces Jacob to try his hand at bungee jumping. Assume Jacob's mass m is suspended in a gravitational field by the bungee of unstretched length b and spring constant k. Besides the longitudinal oscillations due to the bungee jump, Jacob also swings with plane pendulum motion in a vertical plane. Use polar coordinates r, ϕ, neglect air drag, and assume that the bungee always is under tension.

a; Derive the Lagrangian

b; Determine Lagrange's equation of motion for angular motion and identify by name the forces contributing to the angular motion.

c; Determine Lagrange's equation of motion for radial oscillation and identify by name the forces contributing to the tension in the spring.

d; Derive the generalized momenta

e; Determine the Hamiltonian and determine Hamilton's equations of motion.

a: Derive the Lagrangian:

This is identical to the spring pendulum problem. The system is holonomic, conservative, and scleronomic. Introduce plane polar coordinates with radial length r and polar angle ϕ as generalized coordinates. The generalized coordinates are related to the cartesian coodinates by

$$\begin{aligned} y &= r\cos\phi \\ x &= r\sin\phi \end{aligned}$$

Therefore the velocities are given by

$$\begin{aligned} \dot{y} &= \dot{r}\cos\phi + r\dot{\phi}\sin\phi \\ \dot{x} &= \dot{r}\sin\phi - r\dot{\phi}\cos\phi \end{aligned}$$

The kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

The gravitational plus spring potential energies are

$$U = -mgr\cos\phi + \frac{k}{2}(r-b)^2$$

where r_0 denotes the rest length of the spring.

The Lagrangian thus equals

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + mgr\cos\phi - \frac{k}{2}(r-b)^2$$

b; Determine Lagrange's equation of motion for angular motion and identify by name the forces contributing to the angular motion;

For the polar angle ϕ, the Lagrange equation $\Lambda_\phi L = 0$ gives

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = -mgr\sin\phi$$

But the angular momentum $p_\phi = mr^2\dot{\phi}$, thus the equation of motion can be written as

$$\dot{p}_\phi = -mgr\sin\phi$$

Alternatively evaluating $\frac{d}{dt}\left(mr^2\dot{\phi}\right)$ gives

$$mr^2\ddot{\phi} = -mgr\sin\phi - 2mr\dot{r}\dot{\phi}$$

The last term in the right-hand side is the Coriolis force caused by the time variation of the pendulum length r.

c; Determine Lagrange's equation of motion for radial oscillation and identify by name the forces contributing to the tension in the spring.

For the radial distance r, the Lagrange equation $\Lambda_r L = 0$ gives

$$m\ddot{r} = mr\dot{\phi}^2 + mg\cos\phi - k\left(r - r_0\right)$$

This equation just equals the tension in the spring, i.e. $F = m\ddot{r}$. The first term on the right-hand side represents the centrifugal radial acceleration, the second term is the component of the gravitational force, and the third term represents Hooke's Law for the spring. For small amplitudes of ϕ the motion appears as a superposition of harmonic oscillations in the r, ϕ plane.

In this example the orthogonal coordinate approach used gave the tension in the spring thus it is unnecessary to repeat this using the Lagrange multiplier approach.

d; Derive the generalized momenta:

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \end{aligned}$$

e; Determine the Hamiltonian and determine Hamilton's equations of motion.

Since the $\frac{\partial L}{\partial t} = 0$ the Hamiltonian is conservative, and since the transformation from rectangular to cylindrical coordinates does not depend explicitly on time, then the Hamiltonian equals the total energy. Thus

$$\begin{aligned} H &= T + U \\ &= \frac{p_\phi^2}{2mr^2} + \frac{p_r^2}{2m} - mgr\cos\phi + \frac{k}{2}\left(r - r_0\right)^2 \end{aligned}$$

Thus Hamilton's equations of motion are given by the canonical equations

$$\dot{p}_\phi = -\frac{\partial H}{\partial \theta} = -mgr\sin\phi \tag{c}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -k\left(r - b\right) + mg\cos\phi + \frac{p_\phi^2}{mr^3} \tag{d}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} \tag{e}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{f}$$

Equations d and e give that

$$\begin{aligned} \dot{p}_r &= m\ddot{r} = -k\left(r - b\right) + mg\cos\phi + \frac{p_\phi^2}{mr^3} \\ &= -k\left(r - b\right) + mg\cos\phi + mr\dot{\phi}^2 \end{aligned}$$