Final Examination

Final examination:

Will be given Sunday 18 December at 1900-2300 hr in **B&L 106.**

Office hours:

Doug Cline; Friday 16 December, 1300-1500 hr in B&L203D

Content of Final:

The final examination will be similar to the mid-term examination in that it will comprise four full multipart questions. It is a closed-book, nominally a 3-hour, exam but extra time is scheduled. A calculator may be needed but emphasis will be on concepts and the equations of motion rather than numerical answers. There will be no trick questions, all straightforward bookwork. The questions will emphasize material covered starting with the introduction of Lagrangian mechanics as discussed below.

Chapters 1, 2: Not explicitly part of the examination although it is assumed that you have a working knowledge of the material in chapter 2.

Chapters 3, 4, 5: There will be no questions explicitly requiring Newtonian mechanics, linear damped oscillations, or non-linear systems. However, it will be assumed that you have a working knowledge of this material.

Chapter 6: Calculus of variations will not be addressed explicitly, but implicitly it will be assumed in use of Lagrange equations, Hamilton's principle, etc.

Chapter 7: Lagrangian dynamics is **very important**. Applications will cover questions featuring constraints that can be solved with generalized coordinates, Lagrange multipliers, or generalized forces. Must know how to use generalized coordinates, derive equations of motion, and derive forces of constraint. Will require ability to solve the equations of motion. You will not be asked to reproduce the derivation of Lagrange equations from either d'Alembert or Hamilton's Principles.

Chapters 8, 9: Hamiltonian mechanics is **very important.** Need to know the symmetries/conservation laws, Noether's theorem, cyclic coordinates, properties of Hamiltonian, Hamilton's equations of motion. You should be able to use Hamiltonian mechanics to solve simple mechanical systems. Use of the Routhian may feature in a subpart of a question.

Chapters 10: Coupled oscillations and normal modes for both discrete and continuous systems will be a **very important** part of the exam. Need to be able to use equations of motion and solve for characteristic frequencies, eigenfunctions and determine the normal modes for 2 and 3 coupled oscillators or the discrete lattice chain. Maximum determinant will be 3x3.

Chapter 11: Wave motion, wave packets, phase and group velocity all are an **important** part of the exam. No question will involve performing a Fourier transform.

Chapter 12: Conservative two-body central forces will be a **very important** part of the examination.

Chapter 13: Motion in non-inertial reference frames is **very important** for the examination.

Chapter 14: Rigid body rotation in **very important** and will be on the final examination.

Chapter 15: Poisson brackets and their use for canonical transformations is **important** and will feature on the examination. Hamilton-Jacobi theory, canonical perturbation theory, and action-angle variables will not be required for the exam.

Chapter 16: Special relativity is **important** and may feature in a question. General relativity will not be included on the examination.

Chapter 17: The transition to quantum mechanics will not be included on the examination.

Useful formulae

Damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

Solution for $\frac{\Gamma}{2} < \omega_o$ is

$$x = Ae^{-\frac{\Gamma t}{2}} \cos(\omega_1 t - \delta)$$

where

$$\omega_1^2 = \omega_0^2 - \frac{\Gamma^2}{4}$$

Sinusoidal-driven damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = A\cos(\omega t)$$

Steady-state solution is

$$x_{ss} = \frac{A}{\sqrt{(\omega_o^2 - \omega^2)^2 + \Gamma^2\omega^2}} \cos(\omega t - \delta)$$

where

$$\tan\delta = \frac{\Gamma\omega}{(\omega_0^2 - \omega^2)}$$

Newton's law of Gravitation

$$\begin{aligned} \overline{\mathbf{F}}_m &= -G\frac{mM}{r^2}\widehat{\mathbf{r}} \\ \overrightarrow{\boldsymbol{\nabla}} \cdot \overrightarrow{\mathbf{g}} &= -4\pi G\rho \end{aligned}$$

Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Lagrange equations with undetermined multipliers

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{k=1}^{m} \lambda_k(t)\frac{\partial g_k}{\partial q_i} \qquad (i = 1, 2, 3, ...s)$$

Generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian

$$H(q_i, p_i, t) = \sum_i p_i\dot{q}_i - L(q_i, \dot{q}_i, t)$$

Hamilton's equations of motion

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Cylindrical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - U(\rho, z, \phi)$$

$$H = T + U = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right) + U(\rho, z, \phi)$$

Spherical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r, \theta, \phi)$$

$$H = T + U = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r, \theta, \phi)\backslash$$

Poisson Brackets

$$[F, G] \equiv \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$\begin{aligned} [F, F] &= 0 \\ [F, G] &= -[G, F] \\ [G, F + Y] &= [G, F] + [G, Y] \\ [G, FY] &= [G, F]\,Y + F\,[G, Y] \\ 0 &= [F, [G, Y]] + [G, [Y, F]] + [Y\,[F, G]] \end{aligned}$$

$$\dot{q}_k = [q_k, H] = \frac{\partial H}{\partial p_k}$$

$$\dot{p}_k = -[p_k, H] = -\frac{\partial H}{\partial q_k}$$

Canonical transformations

$$\mathcal{H}(Q, P, t) = H(q, p, t) + \frac{\partial F}{\partial t}$$

Generating function	Generating function derivatives	Trivial special case
$F = F_1(\mathbf{q}, \mathbf{Q}, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$ $\quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$ $\quad Q_i = p_i$ $\quad P_i = -q_i$
$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q}\cdot\mathbf{P}$	$p_i = \frac{\partial F_2}{\partial q_i}$ $\quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$ $\quad Q_i = q_i$ $\quad P_i = p_i$
$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \mathbf{q}\cdot\mathbf{p}$	$q_i = -\frac{\partial F_3}{\partial p_i}$ $\quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$ $\quad Q_i = -q_i$ $\quad P_i = -p_i$
$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q}\cdot\mathbf{p} - \mathbf{Q}\cdot\mathbf{P}$	$q_i = -\frac{\partial F_4}{\partial p_i}$ $\quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$ $\quad Q_i = p_i$ $\quad P_i = -q_i$

Hamilton-Jacobi equation

$$H(q; \frac{\partial S}{\partial q}; t) + \frac{\partial S}{\partial t} = 0$$

Orbit differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u})$$

Virial Theorem

$$\langle T\rangle = -\frac{1}{2}\left\langle \sum_i \overrightarrow{\mathbf{F}}_i \cdot \overrightarrow{\mathbf{r}}_i \right\rangle$$

Effective force in rotating reference frame

$$\overrightarrow{\mathbf{F}_{eff}} = m\overrightarrow{\mathbf{a}''} = \overrightarrow{\mathbf{F}} - m\left(\overrightarrow{\mathbf{A}} + 2\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{v}''} + \overrightarrow{\boldsymbol{\omega}} \times \left(\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{r}'}\right) + \dot{\overrightarrow{\boldsymbol{\omega}}} \times \overrightarrow{\mathbf{r}'}\right)$$

Transformation from fixed to rotating frame

$$\left(\frac{d\overrightarrow{\mathbf{G}}}{dt}\right)_{fixed} = \left(\frac{d\overrightarrow{\mathbf{G}}}{dt}\right)_{rotating} + \overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{G}}$$

Inertia tensor

$$I_{ij} \equiv \sum_{\alpha}^{N} m_\alpha \left[\delta_{ij}\left(\sum_{k}^{3} x_{\alpha,k}^2\right) - x_{\alpha,i}x_{\alpha,j}\right]$$

$$I_{ij} = \int \rho\left(\mathbf{r}'\right)\left(\delta_{ij}\left(\sum_{k}^{3} x_k^2\right) - x_i x_j\right) dV$$

Angular momentum

$$\overrightarrow{\mathbf{L}} = \sum_{i}^{n} \overrightarrow{\mathbf{L}_i} = \sum_{i}^{n} \overrightarrow{\mathbf{r}_i} \times \overrightarrow{\mathbf{p}_i}$$

$$\overrightarrow{\mathbf{L}} = \{\mathbf{I}\} \cdot \overrightarrow{\boldsymbol{\omega}}$$

$$L_i = \sum_{j}^{3} I_{ij}\omega_j$$

Parallel-axis theorem

$$J_{ij} \equiv I_{ij} + M\left(a^2\delta_{ij} - a_i a_j\right)$$

Euler equations for rigid body

$$\begin{aligned} N_1^{ext} &= I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 \\ N_2^{ext} &= I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 \\ N_3^{ext} &= I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 \end{aligned}$$

Angular velocity in body-fixed frame

$$\begin{aligned} \omega_1 &= \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}$$

Angular velocity in space-fixed frame

$$\begin{aligned} \omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \omega_z &= \dot{\phi} + \dot{\psi}\cos\theta \end{aligned}$$

Coupled oscillators

$$T = \frac{1}{2}\sum_{j,k} T_{jk}\dot{q}_j\dot{q}_k$$

$$U = \frac{1}{2}\sum_{j,k} V_{jk}q_j q_k$$

$$\sum_j \left(V_{jk} - \omega_r^2 T_{jk}\right) a_{jr} = 0$$

$$T_{jk} \equiv \sum_\alpha m_\alpha \sum_i \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}$$

$$V_{jk} \equiv \left(\frac{\partial^2 U}{\partial q_j \partial q_k}\right)_0$$

Special Relativity

$$\begin{aligned}
\mathbf{p} &= \gamma m \mathbf{u} \\
\gamma &\equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \\
E &= \gamma m c^2 \\
E_0 &= mc^2 \\
E &= T + E_0 \\
E^2 &= p^2c^2 + m^2c^4 \\
L &= -mc^2\sqrt{1-\beta^2} - U \\
H &= T + U + E_0
\end{aligned}$$

Vectors;

$$\overline{\mathbf{C}} = \overline{\mathbf{A}} \times \overline{\mathbf{B}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{A}} = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) \cdot \overrightarrow{\mathbf{C}}$$

$$\overrightarrow{\mathbf{A}} \times \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{C}}\right) - \left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}}\right)\overrightarrow{\mathbf{C}}$$