Classical Mechanics - Module 12

1. Explain what is meant by the following statement: "Lorentz transformations are orthogonal transformations in Minkowski space."
2. Which of the following are invariant quantities in spacetime?
   (a) Energy
   (b) Momentum
   (c) Mass
   (d) Force
   (e) Charge
   (f) The length of a vector
   (g) The length of a four-vector
3. What does it mean for two events to have a spacelike interval? What does it mean for them to have a timelike interval? Draw a picture to support your answer. In which case can events be causally connected?
4. Compare the Lagrangian formalism and the Hamiltonian formalism by creating a two-column chart. Label one side "Lagrangian" and the other side "Hamiltonian" and discuss the similarities and differences. Here are some ideas to get you started:
   - What are the basic variables in each formalism?
   - What are the form and number of the equations of motion derived in each case?
   - How does the Lagrangian "state space" compare to the Hamiltonian "phase space"?
5. Poisson brackets are a powerful means of elucidating when observables are constant of motion and whether two observables can be simultaneously measured with unlimited precision. Consider a spherically symmetric Hamiltonian

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + U(r)$$

   for a mass m where $U(r$ is a central potential. Use the Poisson bracket plus the time dependence to determine the following:

   (a) Does p_ϕ commute with H and is it a constant of motion?
   (b) Does $p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$ commute with H and is it a constant of motion?
   (c) Does p_r commute with H and is it a constant of motion?
   (d) Does p_ϕ commute with p_θ and what does the result imply?

Classical Mechanics – Module 12 Solutions

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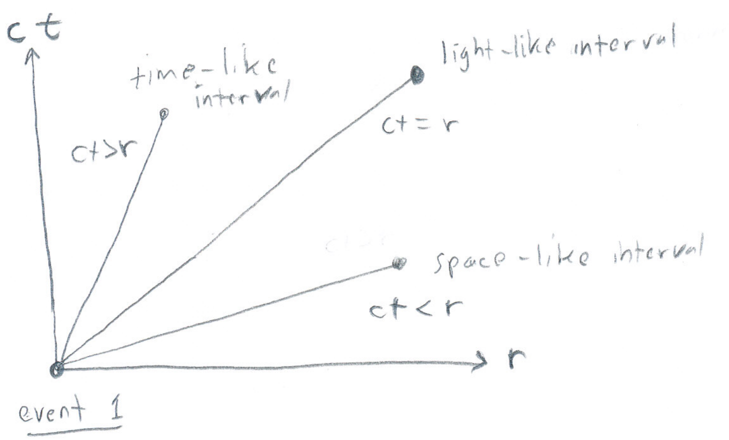
1. A matrix operator, acting on a vector, is orthogonal if it preserves the magnitude of the vector. In Minkowski space, Lorentz transformations correspond to space-time interval preserving rotations. The absolute value of a spacetime interval is defined as $s \equiv \sqrt{x^2 + y^2 + z^2 - c^2t^2}$, and is invariant under Lorentz transformations.

2. Only 4-momentum, charge, and lengths of four-vectors are invariants in spacetime. Energy, mass, and 3-D vectors (including momentum, position, and force) are all frame-dependent, or in other words, depend on the relative velocities of the observer and the frame attached to the quantity being measured. The mass, m, is constant in the rest frame of the particle—but varies depending on your frame of reference.

3. Two events are separated by a space-like (4 vector) interval if the distance between them, $s \equiv \sqrt{x^2 + y^2 + z^2 - c^2t^2} > 0$. For the two events to be separated by a time-like 4-vector, the vector must satisfy $s \equiv \sqrt{x^2 + y^2 + z^2 - c^2t^2} < 0$. Note that the time between the two events, as well as the distance between the two events, must be measured from an inertial reference frame.
Two events must be time-like or light-like to be causally-connected, otherwise information from one event cannot reach the other event! One can represent time-like, space-like, as well as light-like 4-vectors by drawing a single axis for absolute spatial distance and a second for time distance. Then a line at 45 degrees from the r-axis represents light-like intervals, lines at angles greater than 45 represent time-like intervals, and lines at angles less than 45 degrees represent space-like intervals.



4. The Lagrangian uses a set of generalized coordinates as basic variables, while the Hamiltonian uses a set of generalized coordinates and generalized momenta. The Lagrangian formalism results in n 2nd order differential equations of motion for n

generalized coordinates, while the Hamiltonian formalism results in 2n 1st order differential equations of motion, one set for the coordinates and one set for the momenta. The state space represents the coordinates and time-derivative of the coordinates, while the phase space represents the coordinates and momenta.

5.

a. $\frac{dp_\phi}{dt} = \frac{\partial p_\phi}{\partial t} + \{p_\phi, H\}$

Since p_ϕ has no explicit time dependence, the partial derivative with respect to time is 0; $\frac{dp_\phi}{dt} = \{p_\phi, H\} = \frac{\partial p_\phi}{\partial r}\frac{\partial H}{\partial p_r} - \frac{\partial p_\phi}{\partial p_r}\frac{\partial H}{\partial r} + \frac{\partial p_\phi}{\partial \theta}\frac{\partial H}{\partial p_\theta} - \frac{\partial p_\phi}{\partial p_\theta}\frac{\partial H}{\partial \theta} + \frac{\partial p_\phi}{\partial \phi}\frac{\partial H}{\partial p_\phi} - \frac{\partial p_\phi}{\partial p_\phi}\frac{\partial H}{\partial \phi}$

All of the terms are zero, due to a lack of explicit dependence on a particular variable. The only way for this to be non-zero is if the last term, $\frac{\partial H}{\partial \phi}$, is non-zero. So p_ϕ commutes with H and is a constant of the motion.

b. Yes—evalute the commutator and it is a constant of the motion. $L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$ is a constant of the motion…i.e. absolute value of angular momentum is conserved.

c. No, No:

$$\frac{dp_r}{dt} = \{p_r, H\} = \frac{\partial p_r}{\partial r}\frac{\partial H}{\partial p_r} - \frac{\partial p_r}{\partial p_r}\frac{\partial H}{\partial r} + \frac{\partial p_r}{\partial \theta}\frac{\partial H}{\partial p_\theta} - \frac{\partial p_r}{\partial p_\theta}\frac{\partial H}{\partial \theta} + \frac{\partial p_r}{\partial \phi}\frac{\partial H}{\partial p_\phi} - \frac{\partial p_r}{\partial p_\phi}\frac{\partial H}{\partial \phi}$$

$$= -\frac{\partial H}{\partial r}$$

This is consistent with the Hamiltonian equations of motion.

d. Yes—write it out. This implies that the two angular momenta are simulatenously observable.