P235 **PRACTICE FINAL EXAMINATION** **Prof Cline**

1900 - 2215 hr, 18 December 2011

THIS IS A CLOSED BOOK EXAM. Show all steps to get full credit. Answer questions 1,2 in Book 1: questions 3, 4, in Book 2.

Book 1

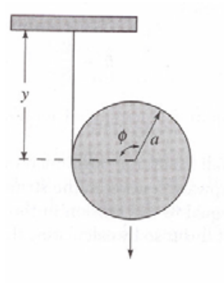
1: [20pts] A yo-yo comprises a circular disk of radius a, thickness b, and mass M that has a string wrapped around the circumference with one end attached to the circumference of the disk and the other end attached to a fixed support. The disk is allowed to fall with the string unwinding as shown. Ignore the mass of the string in the problem.

a) Derive the Lagrangian for the system.

b) Derive the equation of constraint

c) Use Lagrange multipliers to derive the equations of motion and the tension in the string

d) Is ϕ a cyclic variable?



2; [25pts] Consider a jack which comprises six identical point masses m attached at the ends of massless rods along orthogonal axes each a distance L from the center of the jack. The jack is spun about the body-fixed $\hat{\mathbf{3}}$ axis with spin $\dot{\psi} = s$, at an angle θ to the vertical $\hat{\mathbf{z}}$, and with the bottom point mass rotating about a stationary location on a horizontal plane. Assume the jack is subject to a downward gravitational field $\mathbf{g}$.

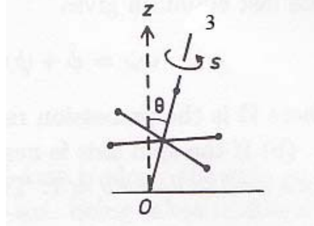
a) Derive the inertia tensor for rotation about one point mass.

b) Derive the Lagrangian in terms of the Euler angles θ, ϕ, ψ where ϕ is the azimuthal angle about the space-fixed $\hat{\mathbf{z}}$ axis and ψ is the corresponding angle about the body-fixed $\hat{\mathbf{3}}$ axis.

c) Determine the angular momentum p_ϕ about the space-fixed $\hat{\mathbf{z}}$ axis, determine if it is conserved?

d) Determine the angular momentum p_ψ about the body-fixed $\hat{\mathbf{3}}$ axis, determine if it is conserved?

e) Express the angular velocity $\dot{\phi}$ about the space-fixed $\hat{\mathbf{z}}$ axis, and $\dot{\psi}$ about the body-fixed $\hat{\mathbf{3}}$ axis in terms of p_ϕ and p_ψ. Can $\dot{\phi}$ and $\dot{\psi}$ be time dependent?



Book 2

3; (25pts) Three identical blocks of mass m ride on a horizontal frictionless linear air track are connected by two identical springs of spring constant κ as shown in the figure.

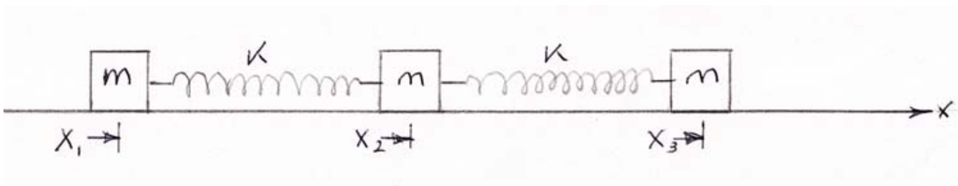
a) Derive the Lagrangian for linear oscillations on the air track.

b) Derive the equations of motion.

c) Derive the eigen frequencies for the system

d) Derive the normal modes of the system

e) What conservation laws apply and what are the implications regarding possible longitudinal oscillatory modes.



4; (30pts) Consider a plane pendulum of length l and mass m that is constrained to swing in a plane which rotates about a vertical axis through the point of support at a constant angular rate ω as determined by some external drive mechanism. The fulcrum of the pendulum is on the vertical axis around which the system is cranked. Assume the instantaneous angle of the pendulum to the vertical axis is θ while the plane of oscillation of the pendulum is at an angle ϕ to the x axis with the angular velocity $\dot{\phi} = \omega$.

a; Derive the Lagrangian

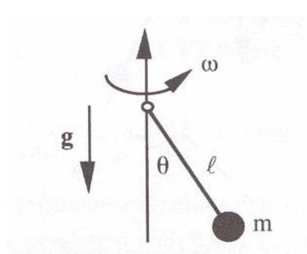
b; Derive the Hamiltonian

c; Derive the equations of motion using Hamiltonian mechanics.

d; Is the non-cyclic Routhian $R = H - p_\phi \dot{\phi}$ a constant of motion, and if so does it equal the total energy?

e; Use the Routhian to derive $\dot{\theta}$ and $\dot{p}_\theta$

f; Use these results to sketch the θ phase-space diagrams when $\omega < \sqrt{\frac{g}{l}}$ and when $\omega > \sqrt{\frac{g}{l}}$. Explain the resultant phase-space diagrams.



Useful formulae

Damped harmonic oscillator:

$$\frac{d^2x}{dt^2}+\Gamma\frac{dx}{dt}+\omega_0^2x=0$$

Solution for $\frac{\Gamma}{2}<\omega_o$ is

$$x=Ae^{-\frac{\Gamma t}{2}}\cos\left(\omega_1 t-\delta\right)$$

where

$$\omega_1^2=\omega_0^2-\frac{\Gamma^2}{4}$$

Sinusoidal-driven damped harmonic oscillator

$$\frac{d^2x}{dt^2}+\Gamma\frac{dx}{dt}+\omega_0^2x=A\cos\left(\omega t\right)$$

Steady-state solution is

$$x_{ss}=\frac{A}{\sqrt{\left(\omega_o^2-\omega^2\right)^2+\Gamma^2\omega^2}}\cos\left(\omega t-\delta\right)$$

where

$$\tan\delta=\frac{\Gamma\omega}{\left(\omega_0^2-\omega^2\right)}$$

Newton's law of Gravitation

$$\begin{aligned}\overline{\mathbf{F}}_m &= -G\frac{mM}{r^2}\widehat{\mathbf{r}}\\ \overrightarrow{\boldsymbol{\nabla}}\cdot\overrightarrow{\mathbf{g}} &= -4\pi G\rho\end{aligned}$$

Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_j}-\frac{\partial L}{\partial q_j}=0$$

Lagrange equations with undetermined multipliers

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}-\frac{\partial L}{\partial q_i}=\sum_{k=1}^{m}\lambda_k\left(t\right)\frac{\partial g_k}{\partial q_i}\qquad\left(i=1,2,3,...s\right)$$

Generalized momentum

$$p_i=\frac{\partial L}{\partial\dot{q}_i}$$

Hamiltonian

$$H\left(q_i,p_i,t\right)=\sum_i p_i\dot{q}_i-L(q_i,\dot{q}_i,t)$$

Hamilton's equations of motion

$$\begin{aligned}\dot{q}_k &= \frac{\partial H}{\partial p_k}\\ -\dot{p}_k &= \frac{\partial H}{\partial q_k}\end{aligned}$$

$$-\frac{\partial L}{\partial t}=\frac{\partial H}{\partial t}$$

Cylindrical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) - U(\rho, z, \phi)$$

$$H = T + U = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right) + U(\rho, z, \phi)$$

Spherical coordinates

$$L = T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r, \theta, \phi)$$

$$H = T + U = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + U(r, \theta, \phi)\backslash$$

Poisson Brackets

$$[F, G] \equiv \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$\begin{aligned} [F, F] &= 0 \\ [F, G] &= -[G, F] \\ [G, F + Y] &= [G, F] + [G, Y] \\ [G, FY] &= [G, F]\, Y + F\, [G, Y] \\ 0 &= [F, [G, Y]] + [G, [Y, F]] + [Y\, [F, G]] \end{aligned}$$

$$\dot{q}_k = [q_k, H] = \frac{\partial H}{\partial p_k}$$

$$\dot{p}_k = -[p_k, H] = -\frac{\partial H}{\partial q_k}$$

Canonical transformations

$$\mathcal{H}(Q, P, t) = H(q, p, t) + \frac{\partial F}{\partial t}$$

Generating function	Generating function derivatives		Trivial special case		
$F = F_1(\mathbf{q}, \mathbf{Q}, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$	$P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$	$Q_i = p_i$	$P_i = -q_i$
$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q} \cdot \mathbf{P}$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$	$Q_i = q_i$	$P_i = p_i$
$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \mathbf{q} \cdot \mathbf{p}$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$	$Q_i = -q_i$	$P_i = -p_i$
$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q} \cdot \mathbf{p} - \mathbf{Q} \cdot \mathbf{P}$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$	$Q_i = p_i$	$P_i = -q_i$

Hamilton-Jacobi equation

$$H(q; \frac{\partial S}{\partial q}; t) + \frac{\partial S}{\partial t} = 0$$

Orbit differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(\frac{1}{u})$$

Virial Theorem

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_i \overrightarrow{\mathbf{F}}_i \cdot \overrightarrow{\mathbf{r}}_i \right\rangle$$

Effective force in rotating reference frame

$$\overrightarrow{\mathbf{F}_{eff}} = m\overrightarrow{\mathbf{a}''} = \overrightarrow{\mathbf{F}} - m\left(\overrightarrow{\mathbf{A}} + 2\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{v}''} + \overrightarrow{\boldsymbol{\omega}} \times \left(\overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{r}'}\right) + \overrightarrow{\dot{\boldsymbol{\omega}}} \times \overrightarrow{\mathbf{r}'}\right)$$

Transformation from fixed to rotating frame

$$\left(\frac{d\overrightarrow{\mathbf{G}}}{dt}\right)_{fixed} = \left(\frac{d\overrightarrow{\mathbf{G}}}{dt}\right)_{rotating} + \overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{G}}$$

Inertia tensor

$$I_{ij} \equiv \sum_{\alpha}^{N} m_\alpha \left[\delta_{ij}\left(\sum_{k}^{3} x_{\alpha,k}^2\right) - x_{\alpha,i}x_{\alpha,j}\right]$$

$$I_{ij} = \int \rho\left(\mathbf{r}'\right)\left(\delta_{ij}\left(\sum_{k}^{3} x_k^2\right) - x_i x_j\right) dV$$

Angular momentum

$$\overrightarrow{\mathbf{L}} = \sum_{i}^{n} \overrightarrow{\mathbf{L}_i} = \sum_{i}^{n} \overrightarrow{\mathbf{r}_i} \times \overrightarrow{\mathbf{p}_i}$$

$$\overrightarrow{\mathbf{L}} = \{\mathbf{I}\} \cdot \overrightarrow{\boldsymbol{\omega}}$$

$$L_i = \sum_{j}^{3} I_{ij}\omega_j$$

Parallel-axis theorem

$$J_{ij} \equiv I_{ij} + M\left(a^2\delta_{ij} - a_i a_j\right)$$

Euler equations for rigid body

$$\begin{aligned}
N_1^{ext} &= I_1\dot{\omega}_1 - \left(I_2 - I_3\right)\omega_2\omega_3 \\
N_2^{ext} &= I_2\dot{\omega}_2 - \left(I_3 - I_1\right)\omega_3\omega_1 \\
N_3^{ext} &= I_3\dot{\omega}_3 - \left(I_1 - I_2\right)\omega_1\omega_2
\end{aligned}$$

Angular velocity in body-fixed frame

$$\begin{aligned}
\omega_1 &= \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\omega_2 &= \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\
\omega_3 &= \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi}
\end{aligned}$$

Angular velocity in space-fixed frame

$$\begin{aligned}
\omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\
\omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\
\omega_z &= \dot{\phi} + \dot{\psi}\cos\theta
\end{aligned}$$

Coupled oscillators

$$T = \frac{1}{2}\sum_{j,k} T_{jk}\dot{q}_j\dot{q}_k$$

$$U = \frac{1}{2}\sum_{j,k} V_{jk}q_jq_k$$

$$\sum_j \left(V_{jk} - \omega_r^2 T_{jk}\right) a_{jr} = 0$$

$$T_{jk} \equiv \sum_\alpha m_\alpha \sum_i \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}$$

$$V_{jk} \equiv \left(\frac{\partial^2 U}{\partial q_j \partial q_k}\right)_0$$

Special Relativity

$$\begin{aligned}
\mathbf{p} &= \gamma m\mathbf{u} \\
\gamma &\equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \\
E &= \gamma mc^2 \\
E_0 &= mc^2 \\
E &= T + E_0 \\
E^2 &= p^2c^2 + m^2c^4 \\
L &= -mc^2\sqrt{1-\beta^2} - U \\
H &= T + U + E_0
\end{aligned}$$

Vectors;

$$\overline{\mathbf{C}}=\overline{\mathbf{A}}\times\overline{\mathbf{B}} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{A}} = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) = 0$$

$$\overrightarrow{\mathbf{A}} \cdot \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \left(\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}\right) \cdot \overrightarrow{\mathbf{C}}$$

$$\overrightarrow{\mathbf{A}} \times \left(\overrightarrow{\mathbf{B}} \times \overrightarrow{\mathbf{C}}\right) = \overrightarrow{\mathbf{B}}\left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{C}}\right) - \left(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}}\right)\overrightarrow{\mathbf{C}}$$