

7:00-10.00pm, 30 October 2009

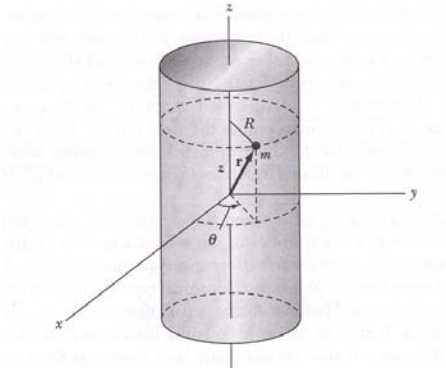
THIS IS A CLOSED BOOK EXAMINATION.

Do all parts of all four questions.
Show all steps to get full credit.

1: (20pts) Consider a rocket fired vertically upwards from the ground in a uniform vertical gravitation field $g = 9.81m/s^2$. The rocket, with initial mass m_I , ejects α kg/sec of propellant at an exhaust velocity u m/s. Derive the equation giving the time dependence of the vertical velocity of the rocket.

2; [30pts] Consider a mass m attracted by a force that is directed toward the origin and proportional to the distance from the origin, $\vec{F} = -k\vec{r}$. The mass is constrained to move on the surface of a cylinder with the origin on the cylindrical axis as shown. The radius of the cylinder is R .

- Derive the Lagrangian and Hamiltonian
- Explain if the Hamiltonian equals the total energy and if it is conserved?
- Derive the equations of motion
- Show which variables, if any, are cyclic
- Derive the frequency of the motion along the axis of the cylinder.



3; (30pts) As part of the Halloween, and the post midterm exam celebrations, Julieta, and Wendi, organize their colleagues to drape crepe ribbons from the highest bridge in Wilson Commons. They plan to do this by dropping rolls of crepe ribbon after fastening the loose end to the parapet. Being very careful physicists they decide that first they need to know the tension in the ribbon to ensure that it does not tear when the roll drops, as well as the speed of the roll after falling $20m$ to ensure it does not hurt anyone. Assume that the rolls of ribbon have a mass M and radius a and that the thickness and mass of the unrolled ribbon is negligible compared with the mass of the roll, that is M and a are constant as it drops, thus the moment of inertia of the roll is $I = \frac{1}{2}Ma^2$.

- Derive the Lagrangian for one roll of crepe dropping with the upper loose end of the crepe ribbon held fixed.
- Use Lagrange multipliers to derive the equations of motion
- Derive the tension in the ribbon.
- Derive the vertical acceleration and final velocity of the roll after falling $20m$ assuming it starts with zero velocity.

4; [30pts] The P235W midterm exam induces Jacob to try his hand at bungee jumping. Assume Jacob's mass m is suspended in a gravitational field by the bungee of unstretched length b and spring constant k . Besides the longitudinal oscillations due to the bungee jump, Jacob also swings with plane pendulum motion in a vertical plane. Use polar coordinates r, ϕ , neglect air drag, and assume that the bungee always is under tension.

- Derive the Lagrangian
- Determine Lagrange's equation of motion for angular motion and identify by name the forces contributing to the angular motion.
- Determine Lagrange's equation of motion for radial oscillation and identify by name the forces contributing to the tension in the spring.
- Derive the generalized momenta
- Determine the Hamiltonian and give all of Hamilton's equations of motion.

Useful formulae

Damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

Solution for $\frac{\Gamma}{2} < \omega_0$ is

$$x = Ae^{-\frac{\Gamma t}{2}} \cos(\omega_1 t - \delta)$$

where

$$\omega_1^2 = \omega_0^2 - \frac{\Gamma^2}{4}$$

Sinusoidal-driven damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 x = A \cos(\omega t)$$

Steady-state solution is

$$x_{ss} = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + \Gamma^2 \omega^2}} \cos(\omega t - \delta)$$

where

$$\tan \delta = \frac{\Gamma \omega}{(\omega_0^2 - \omega^2)}$$

Newton's law of Gravitation

$$\begin{aligned} \vec{\mathbf{F}}_m &= -G \frac{mM}{r^2} \hat{\mathbf{r}} \\ \vec{\nabla} \cdot \vec{\mathbf{g}} &= -4\pi G \rho \end{aligned}$$

Virial Theorem

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_i \vec{\mathbf{F}}_i \cdot \vec{\mathbf{r}}_i \right\rangle$$

Lagrange equations

Generalized coordinates

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Lagrange equations with undetermined multipliers and generalized forces

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{EX} + \sum_{k=1}^m \lambda_k(t) \frac{\partial g_k}{\partial q_j} \quad (i = 1, 2, 3, \dots, s)$$

Generalized momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Hamiltonian

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

Hamilton's equations of motion

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \\ -\frac{\partial L}{\partial t} &= \frac{\partial H}{\partial t} \end{aligned}$$

Cylindrical coordinates

$$\begin{aligned} L &= T - U \\ &= \frac{m}{2} \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \right) - U(\rho, z, \phi) \end{aligned}$$

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2m} \left(p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2 \right) + U(\rho, z, \phi) \end{aligned}$$

Spherical coordinates

$$\begin{aligned} L &= T - U \\ &= \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - U(r, \theta, \phi) \end{aligned}$$

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + U(r, \theta, \phi) \end{aligned}$$

Coupled oscillators

$$T = \frac{1}{2} \sum_{j,k} T_{jk} \dot{q}_j \dot{q}_k$$

$$U = \frac{1}{2} \sum_{j,k} V_{jk} q_j q_k$$

$$T_{jk} \equiv \sum_{\alpha} m_{\alpha} \sum_i \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k}$$

$$V_{jk} \equiv \left(\frac{\partial^2 U}{\partial q_j \partial q_k} \right)_0$$

$$\sum_j (V_{jk} - \omega_r^2 T_{jk}) a_{jr} = 0$$

Vectors;

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{A} = 0$$

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B}) \vec{C}$$